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Optimal Gaussian Sobolev embeddings ☆

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Abstract

A reduction theorem is established, showing that any Sobolev inequality, involving arbitrary rearrangement-invariant norms with respect to the Gauss measure in $\mathbb{R}^n$, is equivalent to a one-dimensional inequality, for a suitable Hardy-type operator, involving the same norms with respect to the standard Lebesgue measure on the unit interval. This result is exploited to provide a general characterization of optimal range and domain norms in Gaussian Sobolev inequalities. Applications to special instances yield optimal Gaussian Sobolev inequalities in Orlicz and Lorentz(–Zygmund) spaces, point out new phenomena, such as the existence of self-optimal spaces, and provide further insight into classical results.

Keywords: Logarithmic Sobolev inequalities; Gauss measure; Sobolev embeddings; Rearrangement-invariant spaces; Optimal domain; Optimal range; Orlicz spaces; Lorentz spaces; Hardy operators involving suprema

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1. Introduction

In connection with the study of quantum fields and hypercontractivity semigroups, extensions of the classical Sobolev inequality in $\mathbb{R}^n$ to the setting when the underlying measure space is infinite-dimensional have been investigated. The main motivation for this research is that, in certain circumstances, the study of quantum fields can be reduced to operator or semigroup estimates which are in turn equivalent to inequalities of Sobolev type in infinitely many variables (see [31] and the references therein).

The classical Sobolev inequality implies that if u is a weakly differentiable function in $\mathbb{R}^n$, decaying to 0 at infinity, and $|\nabla u|^p$ is integrable on $\mathbb{R}^n$ for some $p \in [1, n)$, then $|u|$ raised to the larger power $\frac{np}{n-p}$ is integrable. When $p > n$ (and the support of u has finite measure), u is in fact essentially bounded. Note, in particular, that the gain in the integrability depends on the dimension n.

In attempting to generalize these results to the case where the underlying space is infinite-dimensional, one immediately meets two problems. First, $\frac{np}{n-p} \to p+$ as $n \to \infty$, so the gain in integrability is apparently being lost. Second, and more serious, the Lebesgue measure on an infinite-dimensional space is meaningless.

These problems were overcome in the fundamental paper by L. Gross [25], where the Lebesgue measure was replaced by the Gauss measure γ_n, defined on $\mathbb{R}^n$ by

$$d\gamma_n(x) = (2\pi)^{-\frac{n}{2}} e^{\frac{-|x|^2}{2}}\, dx. \tag{1.1}$$

Since $\gamma_n(\mathbb{R}^n) = 1$ for every $n \in \mathbb{N}$, the extension as $n \to \infty$ is meaningful. The idea was then to seek a version of the Sobolev inequality that would hold on the probability space $(\mathbb{R}^n, \gamma_n)$ with a constant independent of n. In [25] an inequality of this kind is proved, which, in particular, entails that

$$\|u - u_{\gamma_n}\|_{L^2 \operatorname{Log} L(\mathbb{R}^n, \gamma_n)} \leqslant C \|\nabla u\|_{L^2(\mathbb{R}^n, \gamma_n)} \tag{1.2}$$

for some absolute constant C and for every weakly differentiable function u making the right-hand side finite. Here, $u_{\gamma_n} = \int_{\mathbb{R}^n} u(x)\, d\gamma_n(x)$, the mean value of u over $(\mathbb{R}^n, \gamma_n)$, and $L^2 \operatorname{Log} L(\mathbb{R}^n, \gamma_n)$ is the Orlicz space of those functions u such that $|u|^2 |\log |u||$ is integrable in $\mathbb{R}^n$ with respect to γ_n. Observe that (1.2) still provides some slight gain in the integrability from $|\nabla u|$ to u, even though it is no longer a power-gain.

Gross' result ignited an extensive research on Sobolev inequalities in the Gauss space, including simplified proofs [2], applications [23,35,40], and extensions to the case when $|\nabla u|$ belongs to a space different from $L^2(\mathbb{R}^n, \gamma_n)$ [3,4,6,5,11,10,18,27,33]. For instance, inequalities for functions with $|\nabla u| \in L^p(\mathbb{R}^n, \gamma_n)$ for $p \in [1, \infty)$ are known [1], and tell us that then $u \in L^p \operatorname{Log} L^{\frac{p}{2}}(\mathbb{R}^n, \gamma_n)$. Interestingly, in contrast to the Euclidean setting, when $|\nabla u|$ enjoys a high degree of integrability, stronger than just a power, there is a loss of integrability from $|\nabla u|$ to u instead of a gain in the Gaussian Sobolev embedding. This happens, in particular, when $|\nabla u|$ is exponentially integrable [9], or essentially bounded [3]: for instance, in the latter case, one can just infer that $u \in \exp L^2(\mathbb{R}^n, \gamma_n)$, the Orlicz space associated with the Young function $e^{t^2} - 1$. This phenomenon can be explained by the rapid decay of the Gauss measure at infinity.

The aim of this paper is to present a comprehensive treatment of optimal Sobolev embeddings in the Gauss space in the general form

$$\|u - u_{\gamma_n}\|_{Y(\mathbb{R}^n,\gamma_n)} \leqslant C\|\nabla u\|_{X(\mathbb{R}^n,\gamma_n)} \tag{1.3}$$

for some constant C and for every $u \in V^1X(\mathbb{R}^n, \gamma_n)$, where $X(\mathbb{R}^n, \gamma_n)$ and $Y(\mathbb{R}^n, \gamma_n)$ are rearrangement-invariant (for short, r.i.) spaces, and $V^1X(\mathbb{R}^n, \gamma_n)$ is the Sobolev-type space built upon $X(\mathbb{R}^n, \gamma_n)$, namely

$$V^1X\big(\mathbb{R}^n, \gamma_n\big) = \big\{u\colon\ u \text{ is a weakly differentiable function in } \mathbb{R}^n \text{ such that } |\nabla u| \in X\big(\mathbb{R}^n, \gamma_n\big)\big\}.$$

Loosely speaking, in an r.i. space the norm of a function depends only on its degree of integrability, namely on the (Gaussian) measure of its level sets. A precise definition is recalled in Section 2, where the necessary prerequisites from the theory of function spaces are collected.

Our approach relies on a reduction theorem (Theorem 3.1, Section 3) showing that inequality (1.3) is completely equivalent to a one-dimensional inequality for a suitable Hardy-type operator, involving the same norms as in (1.3), but on the interval (0, 1) endowed with the standard Lebesgue measure. This step requires a symmetrization argument exploiting a general Pólya–Szegö principle on the decrease of r.i. norms of the gradient of Sobolev functions in the Gauss space (Theorem 3.2, Section 3), extending the results of [22] and [38]. Its proof relies upon the Gaussian isoperimetric inequality by Borell [11].

The reduction theorem is a key step in our description of the optimal r.i. spaces $X(\mathbb{R}^n, \gamma_n)$ and $Y(\mathbb{R}^n, \gamma_n)$ appearing in (1.3). Namely, given $X(\mathbb{R}^n, \gamma_n)$, we characterize the optimal, i.e. the smallest, range space $Y(\mathbb{R}^n, \gamma_n)$ for which (1.3) holds (Theorem 4.1, Section 4), and, conversely, given $Y(\mathbb{R}^n, \gamma_n)$, we characterize the optimal, i.e. the largest, domain space $X(\mathbb{R}^n, \gamma_n)$ for which (1.3) holds (Theorem 4.3, Section 4).

These results are then employed to establish Sobolev inequalities for concrete spaces. On the one hand, we recover the embeddings mentioned above, corresponding to the choice $X(\mathbb{R}^n, \gamma_n) = L^p(\mathbb{R}^n, \gamma_n)$, with $p \in [1, \infty]$ or $X(\mathbb{R}^n, \gamma_n) = \exp L^\beta(\mathbb{R}^n, \gamma_n)$, with $\beta \in (0, \infty)$, and, as a new contribution, we show their sharpness in the framework of all r.i. spaces.

On the other hand, and more significantly, we establish new embeddings which involve important customary spaces. Section 5 deals with Gaussian Sobolev inequalities in Orlicz spaces. In Theorem 5.1 of that section we associate with any Young function A another Young function A_G such that $L^{A_G}(\mathbb{R}^n, \gamma_n)$ is the optimal Orlicz space in the inequality

$$\|u - u_{\gamma_n}\|_{L^{A_G}(\mathbb{R}^n,\gamma_n)} \leqslant C\|\nabla u\|_{L^A(\mathbb{R}^n,\gamma_n)}$$

for some absolute constant C and for every $u \in V^1L^A(\mathbb{R}^n, \gamma_n)$.

Sobolev embeddings involving Lorentz spaces are the concern of the subsequent Section 6. In fact, Theorem 6.1 deals with the more general class of Lorentz–Zygmund spaces, which naturally come into play when looking for the optimal range or domain in the Gaussian Sobolev inequality.

Finally, in Section 7, a particular feature of Gaussian Sobolev embeddings is pointed out. Indeed, we show that there exist borderline spaces $X(\mathbb{R}^n, \gamma_n)$ which are self-optimal in (1.3), in the sense that (1.3) holds with $Y(\mathbb{R}^n, \gamma_n) = X(\mathbb{R}^n, \gamma_n)$, and the latter is simultaneously the optimal range on the left-hand side and the optimal domain on the right-hand side – see Theorem 7.1. In particular, this is the case when $X(\mathbb{R}^n, \gamma_n) = L^A(\mathbb{R}^n, \gamma_n)$ and A is a Young function given by

$A(t) = e^{\frac{1}{4}(\log t)^2}$ for large t (Corollary 7.2). In fact, it turns out that $L^{A_G}(\mathbb{R}^n, \gamma_n) = L^A(\mathbb{R}^n, \gamma_n)$ for this choice of A.

Part of the results of the present paper were announced in the survey [34, Section 9].

2. Rearrangements and rearrangement-invariant spaces

This section contains the basic background from the theory of rearrangements and of r.i. spaces that will be needed in what follows. For an exhaustive treatment of these topics, we refer the reader to [8]. Definitions and basic properties of the spaces which will play a role in our discussion, such as Orlicz, Lorentz and Lorentz–Zygmund spaces, are also recalled below.

Let (S, m) be a probability space, namely, a measure space S endowed with a probability measure m. We shall assume throughout that (S, m) is totally σ-finite and that m is non-atomic. In fact, S will either be $\mathbb{R}^n$ endowed with the Gaussian measure γ_n, or $(0, 1)$ endowed with the Lebesgue measure. We shall simply write S instead of (S, m) when no ambiguity can arise. We denote by $\mathcal{M}(S)$ the set of real-valued, m-measurable functions on S, and by $\mathcal{M}_+(S)$ the set of nonnegative functions in $\mathcal{M}(S)$.

Let $\phi \in \mathcal{M}(S)$. The *decreasing rearrangement* $\phi^* : (0, 1) \to [0, \infty)$ of ϕ is given by

$$\phi^*(s) = \sup\bigl\{t \geqslant 0\colon\ \gamma_n\bigl(\bigl\{x \in S\colon\ \bigl|\phi(x)\bigr| > t\bigr\}\bigr) > s\bigr\} \quad \text{for } s \in (0, 1).$$

Similarly, the *signed decreasing rearrangement* $\phi^\circ : (0, 1) \to \mathbb{R}$ of ϕ is defined as

$$\phi^\circ(s) = \sup\bigl\{t \in \mathbb{R}\colon\ \gamma_n\bigl(\bigl\{x \in S\colon\ \phi(x) > t\bigr\}\bigr) > s\bigr\} \quad \text{for } s \in (0, 1).$$

We also define $\phi^{**} : (0, 1) \to [0, \infty)$ as

$$\phi^{**}(s) = \frac{1}{s}\int_0^s \phi^*(r)\,dr \quad \text{for } s \in (0, 1).$$

Note that ϕ^{**} is also non-increasing, and $\phi^*(s) \leqslant \phi^{**}(s)$ for $s \in (0, 1)$. Moreover,

$$(\phi + \psi)^{**}(s) \leqslant \phi^{**}(s) + \psi^{**}(s) \quad \text{for } s \in (0, 1), \tag{2.1}$$

for every $\phi, \psi \in \mathcal{M}(S)$.

Two measurable functions ϕ and ψ on S are said to be *equimeasurable* (or *equidistributed*) if $\phi^* = \psi^*$. We shall write

$$\phi \sim \psi$$

to denote that ϕ and ψ are equimeasurable.

A Banach space $X(S)$ of functions in $\mathcal{M}(S)$, equipped with the norm $\|\cdot\|_{X(S)}$, is said to be a *rearrangement-invariant space* if the following five axioms hold:

(P1) $0 \leqslant \psi \leqslant \phi$ a.e. implies $\|\psi\|_{X(S)} \leqslant \|\phi\|_{X(S)}$;
(P2) $0 \leqslant \phi_k \nearrow \phi$ a.e. implies $\|\phi_k\|_{X(S)} \nearrow \|\phi\|_{X(S)}$ as $k \to \infty$;
(P3) $\|1\|_{X(S)} < \infty$;

(P4) a constant C exists such that $\int_S |\phi|\, dm(x) \leqslant C\|\phi\|_{X(S)}$ for every $\phi \in X(S)$;
(P5) $\|\phi\|_{X(S)} = \|\psi\|_{X(S)}$ whenever $\phi^* = \psi^*$.

A norm $\|\cdot\|_{X(S)}$ fulfilling (P1)–(P5) is called an r.i. norm.

A consequence of *Hardy's lemma* [8, Chapter 2, Proposition 3.6 and Theorem 4.6] entails that if $X(S)$ is any r.i. space and $\phi, \psi \in \mathcal{M}(S)$ are measurable functions in S, then

$$\phi^{**}(s) \leqslant \psi^{**}(s) \quad \text{for } s \in (0,1) \quad \text{implies that } \|\phi\|_{X(S)} \leqslant \|\psi\|_{X(S)}. \tag{2.2}$$

We shall also make frequent use of the *Hardy–Littlewood inequality* [8, Chapter 2, Theorem 2.2], which states that

$$\int_S \big|\phi(x)\psi(x)\big|\, dm(x) \leqslant \int_0^1 \phi^*(s)\psi^*(s)\, ds \tag{2.3}$$

for every $\phi, \psi \in \mathcal{M}(S)$.

Given an r.i. space $X(S)$, the set

$$X'(S) = \left\{\phi \in \mathcal{M}(S)\colon \int_S \big|\phi(x)\psi(x)\big|\, dm(x) < \infty \text{ for every } \psi \in X(S)\right\},$$

equipped with the norm

$$\|\phi\|_{X'(S)} = \sup_{\|\psi\|_{X(S)} \leqslant 1} \int_S \big|\phi(x)\psi(x)\big|\, dm(x), \tag{2.4}$$

is called the *associate space* of $X(S)$. It turns out that $X'(S)$ is again an r.i. space endowed with the norm given by (2.4), and that $X''(S) = X(S)$. Furthermore, the Hölder inequality

$$\int_S \big|\phi(x)\psi(x)\big|\, dm(x) \leqslant \|\phi\|_{X(S)}\|\psi\|_{X'(S)} \tag{2.5}$$

holds for every $\phi \in X(S)$ and $\psi \in X'(S)$.

Let $X(S)$ and $Y(S)$ be r.i. spaces. We write $X(S) \to Y(S)$ to denote that $X(S)$ is continuously embedded into $Y(S)$. By [8, Chapter 1, Theorem 1.8],

$$X(S) \subset Y(S) \quad \text{if and only if} \quad X(S) \to Y(S).$$

Moreover,

$$X(S) \to Y(S) \quad \text{if and only if} \quad Y'(S) \to X'(S), \tag{2.6}$$

with the same embedding constants.

For each r.i. space $X(S)$, there exists a unique r.i. space $\overline{X}(0,1)$ on $(0,1)$ satisfying

$$\|\phi\|_{X(S)} = \|\phi^*\|_{\overline{X}(0,1)} \quad \text{for } \phi \in X(S), \tag{2.7}$$

and hence also

$$\|\phi\|_{X(S)} = \|\phi^\circ\|_{\overline{X}(0,1)} \quad \text{for } \phi \in X(S). \tag{2.8}$$

Such a space, endowed with the norm defined by

$$\|f\|_{\overline{X}(0,1)} = \sup_{\|\psi\|_{X'(S)} \leqslant 1} \int_0^1 f^*(s)\psi^*(s)\,ds$$

for $f \in \mathcal{M}(0,1)$, is called the *representation space* of $X(S)$.

Let $X(S)$ be an r.i. space. Then, the function $\varphi_X : [0,1) \to [0,\infty)$ given by

$$\varphi_X(s) = \|\chi_{(0,s)}\|_{\overline{X}(0,1)} \quad \text{for } s \in [0,1),$$

is called the *fundamental function* of $X(S)$. The fundamental function φ_X of any r.i. space $X(S)$ is *quasiconcave*, in the sense that it is non-decreasing on $[0,1)$, $\varphi_X(0) = 0$ and $\frac{\varphi_X(s)}{s}$ is non-increasing on $(0,1)$. Moreover, one has that

$$\varphi_X(s)\varphi_{X'}(s) = s \quad \text{for } s \in [0,1). \tag{2.9}$$

In the remaining part of this section, we recall a few definitions and basic properties of those function spaces that will be involved in our results.

The Lebesgue spaces $L^p(S)$, with $p \in [1,\infty]$, endowed with the standard norm, are the simplest instance of r.i. spaces. In particular, $L^1(S)$ and $L^\infty(S)$ are the largest and the smallest, respectively, r.i. spaces on S, in the sense that if $X(S)$ is any other r.i. space, then

$$L^\infty(S) \to X(S) \to L^1(S).$$

Given any *Young function* $A : [0,\infty) \to [0,\infty)$, namely a convex function vanishing at 0, the *Orlicz space* $L^A(S)$ is the r.i. space of all functions $\phi \in \mathcal{M}(S)$ such that the *Luxemburg norm*

$$\|\phi\|_{L^A(S)} = \inf\left\{\lambda > 0\colon \int_S A\left(\frac{|\phi(x)|}{\lambda}\right) dm(x) \leqslant 1\right\} \tag{2.10}$$

is finite. The function $\widetilde{A} : [0,\infty) \to [0,\infty)$, defined by

$$\widetilde{A}(t) = \sup\{st - A(s)\colon s \geqslant 0\} \quad \text{for } t \in [0,\infty),$$

is also a Young function, called the *Young conjugate* of A. The Orlicz space $L^{\widetilde{A}}(S)$ can be equivalently renormed to become the associate space of $L^A(S)$. In particular, one has that

$$\int_S \left|\phi(x)\psi(x)\right| dm(x) \leqslant 2\|\phi\|_{L^A(S)}\|\psi\|_{L^{\widetilde{A}}(S)} \tag{2.11}$$

for every $\phi \in L^A(S)$ and $\psi \in L^{\widetilde{A}}(S)$.

Since $m(S)<\infty$, one has that $L^A(S)=L^B(S)$ (up to equivalent norms) if and only if A and B are Young functions *equivalent near infinity*, in the sense that $A(C_1 t)\leqslant B(t)\leqslant A(C_2 t)$ for some constants C_1 and C_2, and for large t.

Classes of Orlicz spaces which will be of particular interest in our applications are the *Zygmund spaces of exponential type* $\exp L^\beta(S)$, with $\beta\in(0,\infty)$, and the *Zygmund spaces of logarithmic type* $L^p(\log L)^\alpha(S)$, with either $p\in(1,\infty)$ and $\alpha\in\mathbb{R}$, or $p=1$ and $\alpha\in[0,\infty)$, which are generated by Young functions equivalent to e^{t^β} and to $t^p(\log t)^\alpha$, respectively, near infinity.

Let us mention that, in Section 5, Orlicz spaces on (possibly unbounded) intervals different from $(0,1)$ will also be considered, for technical reasons. The definition of the corresponding Luxemburg norm is then completely analogous.

Let $p\in(0,\infty]$ and let $\omega\in\mathcal{M}_+(0,1)$. Then the *classical Lorentz spaces* $\Lambda^p(\omega)(S)$ and $\Gamma^p(\omega)(S)$ are defined as the sets of those functions $\phi\in\mathcal{M}(S)$ such that the quantities

$$\|\phi\|_{\Lambda^p(\omega)(S)}=\begin{cases}(\int_0^1\phi^*(s)^p\omega(s)\,ds)^{\frac{1}{p}} & \text{if } p\in(0,\infty),\\ \operatorname{ess\,sup}_{0<s<1}\phi^*(s)\omega(s) & \text{if } p=\infty,\end{cases}$$

and

$$\|\phi\|_{\Gamma^p(\omega)(S)}=\begin{cases}(\int_0^1\phi^{**}(s)^p\omega(s)\,ds)^{\frac{1}{p}} & \text{if } p\in(0,\infty),\\ \operatorname{ess\,sup}_{0<s<1}\phi^{**}(s)\omega(s) & \text{if } p=\infty,\end{cases}$$

respectively, are finite. Clearly, one always has $\Gamma^p(\omega)(S)\subset\Lambda^p(\omega)(S)$, and for some p and ω this inclusion may be strict (see [15] and the references therein).

In the case when $p=\infty$, the spaces $\Lambda^\infty(\omega)(S)$ and $\Gamma^\infty(\omega)(S)$ are usually called *Marcinkiewicz spaces*.

It should be noted that, for general p and ω, the sets $\Lambda^p(\omega)(S)$ and $\Gamma^p(\omega)(S)$ need not be r.i. spaces (see [19]). In fact, they may even reduce to the trivial space containing only the zero function.

The quantity $\|\cdot\|_{\Lambda^p(\omega)(S)}$ is equivalent to an r.i. norm, under which $\Lambda^p(\omega)(S)$ is an r.i. space if and only if either $p\in(1,\infty)$ and

$$s^p\int_s^1 r^{-p}\omega(r)\,dr\leqslant C\int_0^s\omega(r)\,dr\quad\text{for } s\in(0,1),$$

or $p=1$ and

$$\frac{1}{s}\int_0^s\omega(\rho)\,d\rho\leqslant\frac{C}{r}\int_0^r\omega(\rho)\,d\rho\quad\text{for } 0<r\leqslant s\leqslant 1,$$

or $p=\infty$ and

$$\frac{1}{s}\int_0^s\frac{dr}{\overline{\omega}(r)}\leqslant\frac{C}{\overline{\omega}(s)}\quad\text{for } s\in(0,1),\tag{2.12}$$

for some constant C, where we have set

$$\overline{\omega}(s) = \operatorname*{ess\,sup}_{0<r<s} \omega(r).$$

Details for the cases where $p \in (1,\infty)$ and $p = 1$ can be found in [36] and [14], respectively. The case where $p = \infty$ follows quite easily from [37, Theorem 3.1].

Important instances of classical Lorentz spaces are the customary two-parameter Lorentz spaces $L^{p,q}(S)$ and $L^{(p,q)}(S)$, defined for $p, q \in (0,\infty]$ as the sets of those functions $\phi \in \mathcal{M}(S)$ for which the quantities

$$\|\phi\|_{L^{p,q}(S)} = \left\| \phi^*(s) s^{\frac{1}{p}-\frac{1}{q}} \right\|_{L^q(0,1)}$$

and

$$\|\phi\|_{L^{(p,q)}(S)} = \left\| \phi^{**}(s) s^{\frac{1}{p}-\frac{1}{q}} \right\|_{L^q(0,1)},$$

respectively, are finite. A generalization is provided by the so-called *Lorentz–Zygmund spaces* $L^{p,q;\alpha}(S)$ and $L^{(p,q;\alpha)}(S)$, which were introduced in [7], and are defined for $p, q \in (0,\infty]$ and $\alpha \in \mathbb{R}$ as the sets of all functions $\phi \in \mathcal{M}(S)$ such that the quantities

$$L^{p,q;\alpha}(S) = \left\| \phi^*(s) s^{\frac{1}{p}-\frac{1}{q}} \bigl(1+\log(1/s)\bigr)^{\alpha} \right\|_{L^q(0,1)}$$

and

$$L^{(p,q;\alpha)}(S) = \left\| \phi^{**}(s) s^{\frac{1}{p}-\frac{1}{q}} \bigl(1+\log(1/s)\bigr)^{\alpha} \right\|_{L^q(0,1)},$$

respectively, are finite. Note that $L^{(p,q;\alpha)}(S) = L^{p,q;\alpha}(S)$ if and only if $p > 1$. Furthermore, $\exp L^{\beta}(S) = L^{\infty,\infty;-\frac{1}{\beta}}(S) = L^{(\infty,\infty;-\frac{1}{\beta})}(S)$ for every $\beta > 0$, and $L^p(\log L)^{\alpha}(S) = L^{p,p;\frac{\alpha}{p}}(S)$ if either $p > 1$ and $\alpha \in \mathbb{R}$, or $p = 1$ and $\alpha \geqslant 0$ (up to equivalent norms).

We recall that $L^{p,q;\alpha}(S)$ is an r.i. space (up to equivalent norms) if and only if one of the following conditions is satisfied:

$$\begin{cases} p = q = 1, \quad \alpha \geqslant 0, \\ 1 < p < \infty, \quad 1 \leqslant q \leqslant \infty, \quad \alpha \in \mathbb{R}, \\ p = \infty, \quad 1 \leqslant q < \infty, \quad \alpha + \dfrac{1}{q} < 0, \\ p = q = \infty, \quad \alpha \leqslant 0 \end{cases} \tag{2.13}$$

(see [7] or [32]). Furthermore,

$$\bigl(L^{p,q;\alpha}\bigr)'(S) = \begin{cases} L^{\infty,\infty;-\alpha}(S) & \text{if } p = q = 1,\ \alpha \geqslant 0, \\ L^{p',q';-\alpha}(S) & \text{if } 1 < p < \infty,\ 1 \leqslant q \leqslant \infty,\ \alpha \in \mathbb{R}, \\ L^{(1,q';-\alpha-1)}(S) & \text{if } p = \infty,\ 1 \leqslant q < \infty,\ \alpha + \frac{1}{q} < 0, \\ L^{1,1;-\alpha}(S) & \text{if } p = q = \infty,\ \alpha \leqslant 0, \end{cases} \tag{2.14}$$

up to equivalent norms. Here, and in what follows, we adopt the usual notation

$$p' = \begin{cases} \infty & \text{if } p = 1, \\ p/(p-1) & \text{if } 1 < p < \infty, \\ 1 & \text{if } p = \infty. \end{cases}$$

We shall also make use of the following characterization of embeddings between Lorentz–Zygmund spaces [7]. Let $p, q_1, q_2 \in [1, \infty]$ and let $\alpha, \beta \in \mathbb{R}$. Then the embedding

$$L^{p,q_1;\alpha}(S) \to L^{p,q_2;\beta}(S)$$

holds if and only if one of the following conditions is satisfied:

$$\begin{cases} 1 \leqslant q_1 \leqslant q_2 \leqslant \infty, \quad p = \infty, \quad \alpha + \dfrac{1}{q_1} \geqslant \beta + \dfrac{1}{q_2}, \\ 1 \leqslant q_1 \leqslant q_2 \leqslant \infty, \quad p < \infty, \quad \alpha \geqslant \beta, \\ 1 \leqslant q_2 < q_1 \leqslant \infty, \quad \alpha + \dfrac{1}{q_1} > \beta + \dfrac{1}{q_2}. \end{cases} \tag{2.15}$$

Let us finally recall that given any quasi-concave, weakly differentiable function $\varphi : [0, 1) \to [0, \infty)$ vanishing at 0, and denoting by φ' its derivative, the spaces $\Lambda^1(\varphi')(S)$ and $\Gamma^\infty(\varphi)(S)$ are r.i. spaces (up to equivalent norms), both with fundamental function φ. They are the smallest and the largest, respectively, r.i. spaces having this fundamental function. Indeed, if $X(S)$ is any other r.i. space with fundamental function $\varphi_X \approx \varphi$, then

$$\Lambda^1(\varphi')(S) \to X(S) \to \Gamma^\infty(\varphi)(S). \tag{2.16}$$

Here and in what follows, the symbol $\approx$ denotes an equivalence up to multiplicative constants. Because of the first embedding in (2.16), the space $\Lambda^1(\varphi')(S)$ is usually called the Lorentz endpoint space corresponding to the fundamental function φ.

If $\overline{\varphi} : (0, 1) \to [0, \infty)$ is the function defined by

$$\overline{\varphi}(s) = \frac{s}{\varphi(s)} \quad \text{for } s \in (0, 1),$$

and $\lim_{s\to 0+} \overline{\varphi}(s) = 0$, then

$$\big(\Lambda^1(\varphi')\big)'(S) = \Gamma^\infty(\overline{\varphi})(S), \qquad \big(\Gamma^\infty(\varphi)\big)'(S) = \Lambda^1(\overline{\varphi}')(S), \tag{2.17}$$

up to equivalent norms.

3. Symmetrization and reduction results

The reduction theorem for the Sobolev inequality (1.3) reads as follows.

Theorem 3.1. *Let $X(\mathbb{R}^n, \gamma_n)$ and $Y(\mathbb{R}^n, \gamma_n)$ be r.i. spaces.*

(i) *If $u \in V^1X(\mathbb{R}^n, \gamma_n)$, then $u \in L^1(\mathbb{R}^n, \gamma_n)$, and, in particular, its mean value u_{γ_n} is well defined.*

(ii) *A constant C_1 exists such that*

$$\|u - u_{\gamma_n}\|_{Y(\mathbb{R}^n,\gamma_n)} \leqslant C_1 \|\nabla u\|_{X(\mathbb{R}^n,\gamma_n)} \tag{3.1}$$

for every $u \in V^1 X(\mathbb{R}^n, \gamma_n)$ if and only if a constant C_2 exists such that

$$\left\| \int_s^1 \frac{f(r)}{r\sqrt{1+\log\frac{1}{r}}}\, dr \right\|_{\overline{Y}(0,1)} \leqslant C_2 \|f\|_{\overline{X}(0,1)} \tag{3.2}$$

for every $f \in \overline{X}(0,1)$. Moreover, C_1 and C_2 depend only on each other.

The proof of Theorem 3.1 relies upon the Pólya–Szegö principle for the Gaussian symmetrization with arbitrary r.i. norms contained in Theorem 3.2 below. Its statement involves the *Gaussian symmetral* $u^\bullet : \mathbb{R}^n \to \mathbb{R}$ of a measurable function u in $\mathbb{R}^n$ defined as

$$u^\bullet(x) = u^\circ\big(\Phi(x_1)\big) \quad \text{for } x = (x_1, \dots, x_n) \in \mathbb{R}^n, \tag{3.3}$$

where $\Phi : \mathbb{R} \to (0,1)$ is the function given by

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-\frac{\tau^2}{2}}\, d\tau \quad \text{for } t \in \mathbb{R}. \tag{3.4}$$

Note that, actually, $u \sim u^\bullet$, since $u \sim u^\circ$, and

$$\Phi(t) = \gamma_n\big(\{x \in \mathbb{R}^n\colon\ x_1 \geqslant t\}\big) \quad \text{for } t \in \mathbb{R}. \tag{3.5}$$

An equivalent formulation of the Pólya–Szegö principle can be given in terms of u° and of the *isoperimetric function* of the Gauss space $I : (0,1) \to (0,\infty)$ defined by

$$I(s) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\Phi^{-1}(s)^2}{2}} \quad \text{for } s \in (0,1), \tag{3.6}$$

and $I(0) = I(1) = 0$. The function I owes its name to the fact that the isoperimetric inequality in the Gauss space reads

$$P_{\gamma_n}(E) \geqslant I\big(\gamma_n(E)\big) \tag{3.7}$$

for every measurable set $E \subset \mathbb{R}^n$ [11], with equality if and only if E is (equivalent to) a half-space [12] (see also [17]). Here,

$$P_{\gamma_n}(E) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\partial^M E} e^{-\frac{|x|^2}{2}}\, d\mathcal{H}^{n-1}(x),$$

the Gaussian perimeter of E, where $\partial^M E$ stands for the essential boundary of E (in the sense of geometric measure theory), and $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. Note that the function I is increasing in $[0, \frac{1}{2}]$, and fulfils

$$I(s) = I(1-s) \quad \text{for } s \in [0,1]. \tag{3.8}$$

Moreover,

$$I(s) \approx s\sqrt{1+\log\frac{1}{s}} \quad \text{for } s \in (0, 1/2], \tag{3.9}$$

with absolute equivalence constants.

Theorem 3.2. *Let $u \in V^1L^1(\mathbb{R}^n, \gamma_n)$. Then u° is locally absolutely continuous in* $(0,1)$. *Moreover, if $X(\mathbb{R}^n, \gamma_n)$ is any r.i. space and $u \in V^1X(\mathbb{R}^n, \gamma_n)$, then $u^\bullet \in V^1X(\mathbb{R}^n, \gamma_n)$ and*

$$\|\nabla u\|_{X(\mathbb{R}^n,\gamma_n)} \geqslant \|\nabla u^\bullet\|_{X(\mathbb{R}^n,\gamma_n)} = \big\|I(s)\big(-u^{\circ\prime}(s)\big)\big\|_{\overline{X}(0,1)}. \tag{3.10}$$

Theorem 3.2 is a straightforward consequence of the following lemma and of property (2.2).

Lemma 3.3. *Let $u \in V^1L(\mathbb{R}^n, \gamma_n)$. Then u° is locally absolutely continuous in* $(0,1)$, *and*

$$\big[I(\cdot)\big(-u^{\circ\prime}(\cdot)\big)\big]^{**}(s) \leqslant |\nabla u|^{**}(s) \quad \textit{for } s \in (0,1). \tag{3.11}$$

Proof. The present proof is reminiscent of arguments from [39] and [16, Theorem 6.5 and Lemma 6.6]. Let $\{(a_k, b_k)\}_{k\in K}$ be a countable family of disjoint intervals $(a_k, b_k) \subset (0,1)$. We have that

$$\begin{aligned}
\int\limits_{\bigcup_{k\in K}\{u^\circ(b_k)<u<u^\circ(a_k)\}} \big|\nabla u(x)\big|\,d\gamma_n(x) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\bigcup_{k\in K}\{u^\circ(b_k)<u<u^\circ(a_k)\}} \big|\nabla u(x)\big| e^{-\frac{x^2}{2}}\,dx \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \sum_{k\in K} \int\limits_{u^\circ(b_k)}^{u^\circ(a_k)} \int\limits_{\partial^M\{u>t\}} e^{-\frac{x^2}{2}}\,d\mathcal{H}^{n-1}(x)\,dt \\
&= \sum_{k\in K} \int\limits_{u^\circ(b_k)}^{u^\circ(a_k)} P_{\gamma_n}\big(\{u>t\}\big)\,dt \\
&\geqslant \sum_{k\in K} \int\limits_{u^\circ(b_k)}^{u^\circ(a_k)} I\big(\gamma_n\big(\{u>t\}\big)\big)\,dt,
\end{aligned} \tag{3.12}$$

where the second equality holds thanks to the coarea formula and the inequality is a consequence of the isoperimetric inequality (3.7). Now, let $0 < \sigma < \frac{1}{2}$ and assume that $[a_k, b_k] \subset [\sigma, 1-\sigma]$ for every $k \in K$. Since

$$a_k \leqslant \gamma_n\big(\{u>t\}\big) \leqslant b_k \quad \text{for } t \in \big(u^\circ(b_k), u^\circ(a_k)\big),$$

inequality (3.12) entails that

$$\int\limits_{\bigcup_{k\in K}\{u^\circ(b_k)<u<u^\circ(a_k)\}} \left|\nabla u(x)\right| d\gamma_n(x) \geqslant I(\sigma)\sum_{k\in K}\bigl(u^\circ(a_k)-u^\circ(b_k)\bigr). \tag{3.13}$$

On the other hand,

$$\gamma_n\Bigl(\bigcup_{k\in K}\bigl\{u^\circ(b_k)<u<u^\circ(a_k)\bigr\}\Bigr)=\sum_{k\in K}\gamma_n\bigl(\bigl\{u^\circ(b_k)<u<u^\circ(a_k)\bigr\}\bigr)\leqslant\sum_{k\in K}(b_k-a_k). \tag{3.14}$$

Thus, inequality (3.13) yields, via the Hardy–Littlewood inequality (2.3),

$$\sum_{k\in K}\bigl(u^\circ(a_k)-u^\circ(b_k)\bigr)\leqslant\frac{1}{I(\sigma)}\int\limits_0^{\sum_{k\in K}(b_k-a_k)}|\nabla u|^*(r)\,dr. \tag{3.15}$$

Owing to the arbitrariness of σ, the local absolute continuity of u° on $(0,1)$ follows, since $|\nabla u|^*\in L^1(0,1)$.

In order to prove (3.11), observe that

$$\begin{aligned}\int\limits_{u^\circ(b_k)}^{u^\circ(a_k)} I\bigl(\gamma_n\bigl(\{u>t\}\bigr)\bigr)\,dt &= \int\limits_{\gamma_n(\{u>u^\circ(a_k)\})}^{\gamma_n(\{u>u^\circ(b_k)\})} I\bigl(\gamma_n\bigl(\bigl\{u>u^\circ(r)\bigr\}\bigr)\bigr)\bigl(-u^{\circ\prime}(r)\bigr)\,dr\\ &=\int\limits_{a_k}^{b_k} I(r)\bigl(-u^{\circ\prime}(r)\bigr)\,dr \quad \text{for } k\in K,\end{aligned} \tag{3.16}$$

where the first equality is a consequence of the (local) absolute continuity of u°, and the second one holds since $\gamma_n(\{u>u^\circ(r)\})=r$ if r does not belong to an interval where u° is constant and $u^{\circ\prime}$ vanishes in any such interval. From (3.12), (3.16) and the Hardy–Littlewood inequality again, we deduce that, for any family of disjoint intervals $\{(a_k,b_k)\}_{k\in K}$ with $(a_k,b_k)\subset(0,1)$,

$$\int\limits_{\bigcup_{k\in K}(a_k,b_k)} I(r)\bigl(-u^{\circ\prime}(r)\bigr)\,dr\leqslant\int\limits_0^{\sum_{k\in K}(b_k-a_k)}|\nabla u|^*(r)\,dr. \tag{3.17}$$

Since each open set in $\mathbb{R}$ is a countable union of disjoint open intervals, inequality (3.17) implies that

$$\int\limits_E I(r)\bigl(-u^{\circ\prime}(r)\bigr)\,dr\leqslant\int\limits_0^{|E|}|\nabla u|^*(r)\,dr \tag{3.18}$$

for every open set $E \subset (0,1)$. In particular, inequality (3.18) tells us that the function $I(r)(-u^{\circ\prime}(r))$ is integrable on $(0,1)$. Thanks to the fact that any measurable set can be approximated from outside by open sets, and thanks to the absolute continuity of the Lebesgue integral, inequality (3.18) continues to hold for any measurable set $E \subset (0,1)$. Hence, (3.11) follows, since

$$\int_0^s \big(I(\cdot)\big(-u^{\circ\prime}(\cdot)\big)\big)^*(r)\,dr = \sup_{|E|=s} \int_E I(r)\big(-u^{\circ\prime}(r)\big)\,dr$$

for $s \in (0,1)$. $\square$

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. (i) By (2.8) and (3.10), it suffices to show that

$$\big\|u^\circ(s) - u^\circ(1/2)\big\|_{L^1(0,1)} \leqslant C\big\|I(s)\big(-u^{\circ\prime}(s)\big)\big\|_{L^1(0,1)} \tag{3.19}$$

for some absolute constant C. By Lemma 3.3, u° is locally absolutely continuous in $(0,1)$. Thus,

$$u^\circ(s) - u^\circ(1/2) = \int_s^{\frac{1}{2}} -u^{\circ\prime}(r)\,dr \quad \text{for } s \in (0,1), \tag{3.20}$$

and hence

$$\begin{aligned}
\big\|u^\circ(s) - u^\circ(1/2)\big\|_{L^1(0,1)} &= \int_0^1 \left| \int_s^{\frac{1}{2}} -u^{\circ\prime}(r)\,dr \right| ds \\
&= \int_0^{\frac{1}{2}} r\big(-u^{\circ\prime}(r)\big)\,dr + \int_{\frac{1}{2}}^1 (1-r)\big(-u^{\circ\prime}(r)\big)\,dr \\
&\leqslant C\left(\int_0^{\frac{1}{2}} I(r)\big(-u^{\circ\prime}(r)\big)\,dr + \int_{\frac{1}{2}}^1 I(1-r)\big(-u^{\circ\prime}(r)\big)\,dr \right) \\
&= C\int_0^1 I(r)\big(-u^{\circ\prime}(r)\big)\,dr = C\big\|I(r)\big(-u^{\circ\prime}(r)\big)\big\|_{L^1(0,1)}
\end{aligned}$$

for some absolute constant C, where the inequality holds owing to (3.9) and the last but one equality owing to (3.8). Hence, (3.19) follows.

(ii) Let us first prove that (3.2) implies (3.1). One has that

$$\|u - u_{\gamma_n}\|_{Y(\mathbb{R}^n,\gamma_n)} \leqslant 2\|u - a\|_{Y(\mathbb{R}^n,\gamma_n)}$$

for any $a \in \mathbb{R}$. Thus, inequality (3.1) will follow if we show that

$$\big\|u - u^\circ(1/2)\big\|_{Y(\mathbb{R}^n,\gamma_n)} \leqslant C\|\nabla u\|_{X(\mathbb{R}^n,\gamma_n)} \tag{3.21}$$

for some constant $C = C(C_2)$ and for every $u \in V^1X(\mathbb{R}^n, \gamma_n)$. By (3.20) and (3.10), inequality (3.21) is in turn reduced to proving that

$$\Bigg\|\int_s^{\frac{1}{2}} f(r)\,dr\Bigg\|_{\overline{Y}(0,1)} \leqslant C\big\|I(s)f(s)\big\|_{\overline{X}(0,1)} \tag{3.22}$$

for some positive constant $C = C(C_2)$ and for every $f \in \overline{X}(0,1)$. By (3.2) applied to $f(s)$ replaced by $\chi_{(0,\frac{1}{2})}(s)s\sqrt{1+\log\frac{1}{s}}f(s)$, and by (3.9), one has that

$$\begin{aligned}\Bigg\|\chi_{(0,\frac{1}{2})}(s)\int_s^{\frac{1}{2}} f(r)\,dr\Bigg\|_{\overline{Y}(0,1)} &\leqslant C_2\Bigg\|\chi_{(0,\frac{1}{2})}(s)s\sqrt{1+\log\frac{1}{s}}f(s)\Bigg\|_{\overline{X}(0,1)}\\ &\leqslant C\big\|\chi_{(0,\frac{1}{2})}(s)I(s)f(s)\big\|_{\overline{X}(0,1)}\end{aligned} \tag{3.23}$$

for some constant $C = C(C_2)$ and for every $f \in \overline{X}(0,1)$. On the other hand, for any such f,

$$\begin{aligned}&\Bigg\|\chi_{(\frac{1}{2},1)}(s)\int_s^{\frac{1}{2}} f(r)\,dr\Bigg\|_{\overline{Y}(0,1)}\\ &\quad= \Bigg\|\chi_{(\frac{1}{2},1)}(s)\int_{1-s}^{\frac{1}{2}} f(1-r)\,dr\Bigg\|_{\overline{Y}(0,1)}\\ &\quad= \Bigg\|\chi_{(\frac{1}{2},1)}(1-s)\int_s^{\frac{1}{2}} f(1-r)\,dr\Bigg\|_{\overline{Y}(0,1)} \qquad \big(\text{since } \|\cdot\|_{\overline{Y}} \text{ is an r.i. norm}\big)\\ &\quad= \Bigg\|\chi_{(0,\frac{1}{2})}(s)\int_s^{\frac{1}{2}} f(1-r)\,dr\Bigg\|_{\overline{Y}(0,1)}\\ &\quad\leqslant C\big\|\chi_{(0,\frac{1}{2})}(s)I(s)f(1-s)\big\|_{\overline{X}(0,1)} \qquad \big(\text{by (3.23)}\big)\\ &\quad= C\big\|\chi_{(0,\frac{1}{2})}(1-s)I(1-s)f(s)\big\|_{\overline{X}(0,1)} \qquad \big(\text{since } \|\cdot\|_{\overline{X}} \text{ is an r.i. norm}\big)\\ &\quad= C\big\|\chi_{(\frac{1}{2},1)}(s)I(s)f(s)\big\|_{\overline{X}(0,1)} \qquad \big(\text{by (3.8)}\big).\end{aligned} \tag{3.24}$$

Combining (3.23) and (3.24) yields (3.22), and hence (3.2).

Let us now prove that (3.2) implies (3.1). Given any locally integrable function $f : (0,1) \to [0,\infty)$ such that

$$f(s) = f(1-s) \quad \text{for } s \in (0,1), \tag{3.25}$$

define $v : \mathbb{R}^n \to \mathbb{R}$ by

$$v(x) = \int_{\Phi(x_1)}^{\frac{1}{2}} \frac{f(r)}{I(r)}\,dr \quad \text{for } x \in \mathbb{R}^n, \tag{3.26}$$

where $\Phi : \mathbb{R} \to [0,1]$ is given by (3.4). Owing to (3.8) and (3.25),

$$v_{\gamma_n} = 0. \tag{3.27}$$

Moreover, by (3.5), we have that

$$v^\circ(s) = \int_s^{\frac{1}{2}} \frac{f(r)}{I(r)}\,dr \quad \text{for } s \in (0,1). \tag{3.28}$$

On the other hand,

$$\bigl|\nabla v(x)\bigr| = \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{x_1^2}{2}}}{I(\Phi(x_1))} f\bigl(\Phi(x_1)\bigr) = f\bigl(\Phi(x_1)\bigr) \quad \text{for a.e. } x \in \mathbb{R}^n, \tag{3.29}$$

where the last equality holds thanks to (3.6). Eq. (3.29) implies that

$$|\nabla v|^*(s) = f^*(s) \quad \text{for } s \in (0,1). \tag{3.30}$$

Owing to (3.27), (3.28) and (3.30), inequality (3.1) applied to $u = v$ implies that

$$\left\| \int_s^{\frac{1}{2}} \frac{f(r)}{I(r)}\,dr \right\|_{\overline{Y}(0,1)} \leqslant C_1 \|f\|_{\overline{X}(0,1)} \tag{3.31}$$

for every f as above. Hence, it is easily seen that

$$\left\| \chi_{(0,\frac{1}{2})}(s) \int_s^{\frac{1}{2}} f(r)\,dr \right\|_{\overline{Y}(0,1)} \leqslant 2C_1 \bigl\| \chi_{(0,\frac{1}{2})}(s) I(s) f(s) \bigr\|_{\overline{X}(0,1)} \tag{3.32}$$

for every locally integrable function $f : (0,1) \to [0,\infty)$. Now, for any such f,

$$\begin{aligned}\left\| \int_s^1 f(r)\,dr \right\|_{\overline{Y}(0,1)} &\leqslant \left\| \int_s^1 f(r)\chi_{(0,\frac{1}{2})}(r)\,dr \right\|_{\overline{Y}(0,1)} + \left\| \int_s^1 f(r)\chi_{(\frac{1}{2},1)}(r)\,dr \right\|_{\overline{Y}(0,1)} \\ &= \left\| \chi_{(0,\frac{1}{2})}(s) \int_s^{\frac{1}{2}} f(r)\,dr \right\|_{\overline{Y}(0,1)} + \left\| \int_s^1 f(r)\chi_{(\frac{1}{2},1)}(r)\,dr \right\|_{\overline{Y}(0,1)} \\ &\leqslant 2C_1 \left\| \chi_{(0,\frac{1}{2})}(s) I(s) f(s) \right\|_{\overline{X}(0,1)} + \int_{\frac{1}{2}}^1 f(r)\,dr \|1\|_{\overline{Y}(0,1)}, \end{aligned} \tag{3.33}$$

where the last inequality holds thanks to (3.32). Since an absolute constant C exists such that

$$\chi_{(0,\frac{1}{2})}(s) I(s) \leqslant Cs\sqrt{1+\log\frac{1}{s}} \quad \text{and} \quad \chi_{(\frac{1}{2},1)}(s) \leqslant Cs\sqrt{1+\log\frac{1}{s}} \quad \text{for } s \in (0,1),$$

the rightmost side of (3.33) does not exceed

$$\bigl(2C_1C + C\|1\|_{\overline{X}'(0,1)}\|1\|_{\overline{Y}(0,1)}\bigr) \left\| s\sqrt{1+\log\frac{1}{s}}\, f(s) \right\|_{\overline{X}(0,1)}.$$

Notice that here we have made use of inequality (2.5). Thus, (3.2) follows. □

4. Optimal range and optimal domain in the Gaussian Sobolev inequality

Let $X(\mathbb{R}^n,\gamma_n)$ and $Y(\mathbb{R}^n,\gamma_n)$ be r.i. spaces. We say that $Y(\mathbb{R}^n,\gamma_n)$ is the *optimal range* for $X(\mathbb{R}^n,\gamma_n)$ in the Gaussian Sobolev inequality (1.3) if:

i) inequality (1.3) holds;
ii) if $Z(\mathbb{R}^n,\gamma_n)$ is an r.i. space such that (1.3) holds with $Y(\mathbb{R}^n,\gamma_n)$ replaced by $Z(\mathbb{R}^n,\gamma_n)$, then $Y(\mathbb{R}^n,\gamma_n) \to Z(\mathbb{R}^n,\gamma_n)$.

Analogously, the space $X(\mathbb{R}^n,\gamma_n)$ is said to be the *optimal domain* for $Y(\mathbb{R}^n,\gamma_n)$ in the Gaussian Sobolev inequality (1.3) if:

i) inequality (1.3) holds;
ii) if $Z(\mathbb{R}^n,\gamma_n)$ is an r.i. space such that (1.3) holds with $X(\mathbb{R}^n,\gamma_n)$ replaced by $Z(\mathbb{R}^n,\gamma_n)$, then $Z(\mathbb{R}^n,\gamma_n) \to X(\mathbb{R}^n,\gamma_n)$.

Finally, we say that $(X(\mathbb{R}^n,\gamma_n), Y(\mathbb{R}^n,\gamma_n))$ is an *optimal pair* in the Gaussian Sobolev inequality (1.3) if $Y(\mathbb{R}^n,\gamma_n)$ is the optimal range for $X(\mathbb{R}^n,\gamma_n)$ and, simultaneously, $X(\mathbb{R}^n,\gamma_n)$ is the optimal domain for $Y(\mathbb{R}^n,\gamma_n)$.

The optimal range in the Gaussian Sobolev inequality (1.3) for a given domain is characterized in the following theorem.

Theorem 4.1. *Let $X(\mathbb{R}^n,\gamma_n)$ be an r.i. space, and let $Y(\mathbb{R}^n,\gamma_n)$ be the r.i. space whose associate norm is given by*

$$\|u\|_{Y'(\mathbb{R}^n,\gamma_n)} = \left\| \frac{u^{**}(s)}{\sqrt{1+\log\frac{1}{s}}} \right\|_{\overline{X}'(0,1)} \tag{4.1}$$

for any $u \in \mathcal{M}(\mathbb{R}^n)$. Then $Y(\mathbb{R}^n,\gamma_n)$ is the optimal range for $X(\mathbb{R}^n,\gamma_n)$ in the Gaussian Sobolev inequality (1.3).

Our discussion of the optimal domain in (1.3) starts with the following lemma, where a somewhat implicit description is provided for any admissible range $Y(\mathbb{R}^n,\gamma_n)$ fulfilling

$$\exp L^2\big(\mathbb{R}^n,\gamma_n\big) \to Y\big(\mathbb{R}^n,\gamma_n\big) \to L(\log L)^{\frac{1}{2}}\big(\mathbb{R}^n,\gamma_n\big). \tag{4.2}$$

Note that assumption (4.2) is natural in this setting, since, as anticipated above, $\exp L^2(\mathbb{R}^n,\gamma_n)$ is the optimal range corresponding to the smallest possible domain $L^\infty(\mathbb{R}^n,\gamma_n)$, and $L(\log L)^{\frac{1}{2}}(\mathbb{R}^n,\gamma_n)$ is the optimal range corresponding to the largest possible domain $L^1(\mathbb{R}^n,\gamma_n)$ (see Proposition 4.4 below).

Lemma 4.2. *Let $Y(\mathbb{R}^n,\gamma_n)$ be an r.i. space satisfying* (4.2). *Define*

$$\|u\|_{X(\mathbb{R}^n,\gamma_n)} = \sup_{0\leqslant h\sim u} \left\| \int_s^1 \frac{h(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{\overline{Y}(0,1)} \tag{4.3}$$

for any $u \in \mathcal{M}(\mathbb{R}^n)$, and let $X(\mathbb{R}^n,\gamma_n)$ be the set of all $u \in \mathcal{M}(\mathbb{R}^n)$ such that $\|u\|_{X(\mathbb{R}^n,\gamma_n)} < \infty$. Then $\|\cdot\|_{X(\mathbb{R}^n,\gamma_n)}$ is an r.i. norm, and hence $X(\mathbb{R}^n,\gamma_n)$ is an r.i. space equipped with this norm. Moreover, $X(\mathbb{R}^n,\gamma_n)$ is the optimal domain for $Y(\mathbb{R}^n,\gamma_n)$ in the Gaussian Sobolev embedding (1.3).

A more explicit characterization of the optimal domain in (1.3) is given in the next theorem under a slight strengthening of the second embedding in (4.2), which holds in customary situations. It amounts to a boundedness property of the supremum-type Hardy operator T defined for $f \in \mathcal{M}(0,1)$ as

$$Tf(s) = \sqrt{1+\log\frac{1}{s}} \sup_{s\leqslant r\leqslant 1} \frac{f^*(r)}{\sqrt{1+\log\frac{1}{r}}} \quad \text{for } s \in (0,1). \tag{4.4}$$

Theorem 4.3. *Let $Y(\mathbb{R}^n,\gamma_n)$ be an r.i. space such that*

$$\exp L^2\big(\mathbb{R}^n,\gamma_n\big) \to Y\big(\mathbb{R}^n,\gamma_n\big) \tag{4.5}$$

and

$$T \text{ is bounded on } \overline{Y}'(0,1). \tag{4.6}$$

Then (4.2) *holds, and the optimal domain* $X(\mathbb{R}^n,\gamma_n)$ *for* $Y(\mathbb{R}^n,\gamma_n)$ *in the Gaussian Sobolev inequality* (1.3) *fulfils*

$$\|u\|_{X(\mathbb{R}^n,\gamma_n)} \approx \left\| \int_s^1 \frac{u^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{\overline{Y}(0,1)} \tag{4.7}$$

for $u \in \mathcal{M}(\mathbb{R}^n)$*, with absolute equivalence constants.*

An application of Theorems 4.1 and 4.3, combined with rather standard Hardy-type inequalities, leads to the following result, dealing with some basic examples. Deeper conclusions derived via Theorems 3.1, 4.1 and 4.3, concerning new sharp Sobolev embeddings, are presented in the last three sections.

Proposition 4.4.

(i) *Let* $p \in [1,\infty)$*. Then a constant* $C = C(p)$ *exists such that*

$$\|u - u_{\gamma_n}\|_{L^p(\log L)^{\frac{p}{2}}(\mathbb{R}^n,\gamma_n)} \leqslant C\|\nabla u\|_{L^p(\mathbb{R}^n,\gamma_n)} \tag{4.8}$$

for every $u \in V^1L^p(\mathbb{R}^n,\gamma_n)$*. Moreover,* $(L^p(\mathbb{R}^n,\gamma_n), L^p(\log L)^{\frac{p}{2}}(\mathbb{R}^n,\gamma_n))$ *is an optimal pair in* (4.8).

(ii) *An absolute constant* C *exists such that*

$$\|u - u_{\gamma_n}\|_{\exp L^2(\mathbb{R}^n,\gamma_n)} \leqslant C\|\nabla u\|_{L^\infty(\mathbb{R}^n,\gamma_n)} \tag{4.9}$$

for every $u \in V^1L^\infty(\mathbb{R}^n,\gamma_n)$*. Moreover,* $(L^\infty(\mathbb{R}^n,\gamma_n), \exp L^2(\mathbb{R}^n,\gamma_n))$ *is an optimal pair in* (4.9).

(iii) *Let* $\beta \in (0,\infty)$*. Then, a constant* $C = C(\beta)$ *exists such that*

$$\|u - u_{\gamma_n}\|_{\exp L^{\frac{2\beta}{2+\beta}}(\mathbb{R}^n,\gamma_n)} \leqslant C\|\nabla u\|_{\exp L^\beta(\mathbb{R}^n,\gamma_n)} \tag{4.10}$$

for every $u \in V^1\exp L^\beta(\mathbb{R}^n,\gamma_n)$*. Moreover,* $(\exp L^\beta(\mathbb{R}^n,\gamma_n), \exp L^{\frac{2\beta}{2+\beta}}(\mathbb{R}^n,\gamma_n))$ *is an optimal pair in* (4.10).

The examples contained in Proposition 4.4 demonstrate the interesting phenomenon to which we alluded in Section 1: while there is a *gain* in integrability when the domain is a Lebesgue space, there is actually a *loss* in integrability when the domain is close to $L^\infty(\mathbb{R}^n,\gamma_n)$ (observe that $\frac{2\beta}{2+\beta} < \beta$ when $\beta > 0$).

We note that the embeddings (4.8)–(4.10) considered in Proposition 4.4 are well known. Our contribution consists in the proof of their optimality. In particular, it follows that in Gross's original result as well as in its later generalizations, both the range and the domain were already sharp in the broad context of r.i. spaces.

Our first concern is to establish Theorem 4.1, Lemma 4.2 and Theorem 4.3; the proof of Proposition 4.4 is postponed to the end of this section.

Proof of Theorem 4.1. First, we claim that the functional defined by (4.1) actually defines an r.i. norm on $(\mathbb{R}^n, \gamma_n)$. To prove this claim, it suffices to show that the functional given by

$$\left\| \frac{f^{**}(s)}{\sqrt{1+\log\frac{1}{s}}} \right\|_{\overline{X}'(0,1)}$$

for any $f \in \mathcal{M}(0,1)$ defines an r.i. norm on $(0,1)$. The positive homogeneity and nontriviality are clear. The triangle inequality follows from the subadditivity of the operation $f \mapsto f^{**}$. Properties (P1) and (P2) are satisfied thanks to standard properties of the decreasing rearrangement (see [8]). Property (P3) is a straightforward consequence of the same property for $\overline{X}'(0,1)$. From property (P4) for $\overline{X}'(0,1)$, we get that

$$\begin{aligned} \left\| \frac{f^{**}(s)}{\sqrt{1+\log\frac{1}{s}}} \right\|_{\overline{X}'(0,1)} &\geqslant f^{**}(1) \left\| \frac{1}{\sqrt{1+\log\frac{1}{s}}} \right\|_{\overline{X}'(0,1)} \geqslant C f^{**}(1) \int_0^1 \frac{ds}{\sqrt{1+\log\frac{1}{s}}} \\ &\geqslant C' \|f\|_{L^1(0,1)}, \end{aligned}$$

for suitable constants C and C' depending on X. This proves (P4). Since (P5) is obvious, our claim follows.

Owing to Theorem 3.1, the Gaussian Sobolev inequality (1.3) holds with $Y(\mathbb{R}^n, \gamma_n)$ as in the statement if (and only if)

$$\left\| \int_s^1 \frac{f(r)}{r\sqrt{1+\log\frac{1}{r}}}\, dr \right\|_{\overline{Y}(0,1)} \leqslant C \|f\|_{\overline{X}(0,1)} \tag{4.11}$$

for some constant C and every $f \in \overline{X}(0,1)$. By the very definition of the associate norm and by Fubini's theorem, we have that

$$\begin{aligned} \sup_{\|f\|_{\overline{X}(0,1)}\leqslant 1} \left\| \int_s^1 \frac{f(r)}{r\sqrt{1+\log\frac{1}{r}}}\, dr \right\|_{\overline{Y}(0,1)} &= \sup_{\|f\|_{\overline{X}(0,1)}\leqslant 1} \sup_{\|g\|_{\overline{Y}'(0,1)}\leqslant 1} \int_0^1 g^*(s) \int_s^1 \frac{|f(r)|}{r\sqrt{1+\log\frac{1}{r}}}\, dr\, ds \\ &= \sup_{\|g\|_{\overline{Y}'(0,1)}\leqslant 1} \sup_{\|f\|_{\overline{X}(0,1)}\leqslant 1} \int_0^1 |f(r)| \frac{g^{**}(r)}{\sqrt{1+\log\frac{1}{r}}}\, dr \\ &= \sup_{\|g\|_{\overline{Y}'(0,1)}\leqslant 1} \left\| \frac{g^{**}(r)}{\sqrt{1+\log\frac{1}{r}}} \right\|_{\overline{X}'(0,1)} = 1. \end{aligned} \tag{4.12}$$

Note that the last equality holds by the definition of the norm in $\|\cdot\|_{\overline{Y}'(0,1)}$. Hence, (4.11) follows.

It remains to show that $Y(\mathbb{R}^n, \gamma_n)$ is the optimal range for $X(\mathbb{R}^n, \gamma_n)$. To this purpose, suppose that $Z(\mathbb{R}^n, \gamma_n)$ is another r.i. space such that

$$\|u - u_{\gamma_n}\|_{Z(\mathbb{R}^n,\gamma_n)} \leqslant C \|\nabla u\|_{X(\mathbb{R}^n,\gamma_n)}$$

for some constant C and every $u \in V^1X(\mathbb{R}^n, \gamma_n)$. By Theorem 3.1 again, this is equivalent to

$$\left\| \int_s^1 \frac{f(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{\overline{Z}(0,1)} \leqslant C\|f\|_{\overline{X}(0,1)}$$

for some constant C and every $f \in \overline{X}(0,1)$. Via a chain analogous to (4.12), one can deduce from this inequality that

$$\left\| \frac{g^{**}(r)}{\sqrt{1+\log\frac{1}{r}}} \right\|_{\overline{X}'(0,1)} \leqslant C\|g\|_{\overline{Z}'(0,1)}$$

for every $g \in \overline{Z}'(0,1)$. The last inequality is equivalent to the embedding $\overline{Z}'(0,1) \to \overline{Y}'(0,1)$, which is in turn equivalent to $Y(\mathbb{R}^n, \gamma_n) \to Z(\mathbb{R}^n, \gamma_n)$. This shows the optimality of $Y(\mathbb{R}^n, \gamma_n)$. $\square$

Let us now come to the proofs concerning the optimal domain in (1.3). We begin with Lemma 4.2.

Proof of Lemma 4.2. We shall prove that the functional $\|u\|_{X(\mathbb{R}^n,\gamma_n)}$ is an r.i. norm. The fact that $X(\mathbb{R}^n, \gamma_n)$ is the optimal domain for $Y(\mathbb{R}^n, \gamma_n)$ in the Gaussian Sobolev embedding (1.3) will then immediately follow via Theorem 3.1.

It suffices to show that the functional $\|f\|_{\overline{X}(0,1)}$ defined for any function $f \in \mathcal{M}(0,1)$ as

$$\|f\|_{\overline{X}(0,1)} = \sup_{0\leqslant h\sim f} \left\| \int_s^1 \frac{h(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{\overline{Y}(0,1)},$$

is an r.i. norm. We begin by showing that $\|\cdot\|_{\overline{X}(0,1)}$ is actually a norm; we shall then prove that it fulfils properties (P1)–(P5) of r.i. norms.

The only nontrivial property of norms to be verified for $\|\cdot\|_{\overline{X}(0,1)}$ is the triangle inequality. To this purpose, let us first observe that, if $f, g \in \mathcal{M}_+(0,1)$ and $f \leqslant g$ a.e. in $(0,1)$, then

$$\|f\|_{\overline{X}(0,1)} \leqslant \|g\|_{\overline{X}(0,1)}. \tag{4.13}$$

Indeed, by [8, Chapter 2, Corollary 7.6], for any nonnegative function $h \sim f$ there exists a measure-preserving map $H : (0,1) \to (0,1)$ such that $h = h^* \circ H = f^* \circ H$. Since $f^* \leqslant g^*$ in $(0,1)$, $h \leqslant g^* \circ H \sim g$, where the equimeasurability of the last two functions holds owing to [8, Chapter 2, Proposition 7.2]. Hence, (4.13) follows.

Next, it is not difficult to show that for any simple functions f, g and h in $(0,1)$ such that $h \sim f + g$, there exist (simple) functions h_f and h_g such that

$$h_f \sim f, \qquad h_g \sim g \quad \text{and} \quad h = h_f + h_g. \tag{4.14}$$

Now, let $f, g \in \mathcal{M}(0,1)$. By a standard result of measure theory there exist sequences of nonnegative simple functions $\{f_k\}$ and $\{g_k\}$ such that

$$f_k \nearrow |f| \quad \text{and} \quad g_k \nearrow |g| \quad \text{as } k \to \infty. \tag{4.15}$$

In particular,

$$\lim_{k\to\infty} (f_k + g_k)^* = \bigl(|f| + |g|\bigr)^* \quad \text{in } (0,1). \tag{4.16}$$

Given any $h \in \mathcal{M}_+(0,1)$ such that

$$h \sim |f| + |g|,$$

there exists a measure-preserving map H such that

$$h = h^* \circ H = \bigl(|f| + |g|\bigr)^* \circ H.$$

Define the sequence $\{h_k\}$ by

$$h_k = (f_k + g_k)^* \circ H \quad \text{for } k \in \mathbb{N}.$$

Thus,

$$h_k \sim f_k + g_k \quad \text{for } k \in \mathbb{N},$$

and

$$\lim_{k\to\infty} h_k = h \quad \text{in } (0,1),$$

by (4.16). Moreover,

$$h_k^{**}(s) = (f_k + g_k)^{**}(s) \leqslant f_k^{**}(s) + g_k^{**}(s) \leqslant f^{**}(s) + g^{**}(s) \quad \text{for } s \in (0,1), \tag{4.17}$$

for $k \in \mathbb{N}$. By (4.17), the functions h_k are equiintegrable in $(0,1)$, and since the function $\frac{1}{r\sqrt{1+\log\frac{1}{r}}}$ is bounded in $(s,1)$ for every $s \in (0,1)$, the functions $\frac{h_k(r)}{r\sqrt{1+\log\frac{1}{r}}}$ are equiintegrable in $(s,1)$ as well. Consequently,

$$\lim_{k\to\infty} \int_s^1 \frac{h_k(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr = \int_s^1 \frac{h(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \quad \text{for } s \in (0,1). \tag{4.18}$$

From (4.18), via the Fatou property of r.i. norms [8, Theorem 1.7, Chapter 1], we deduce that

$$\left\| \int_s^1 \frac{h(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{\overline{Y}(0,1)} \leqslant \liminf_{k\to\infty} \left\| \int_s^1 \frac{h_k(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{\overline{Y}(0,1)}. \tag{4.19}$$

By property (4.14), there exist two sequences of functions $\{h_{f_k}\}$ and $\{h_{g_k}\}$ such that

$$h_{f_k} \sim f_k, \qquad h_{g_k} \sim g_k \quad \text{and} \quad h_k = h_{f_k} + h_{g_k} \quad \text{for } k \in \mathbb{N}.$$

On the other hand, there exist two sequences of measure-preserving maps $\{H_{f_k}\}$ and $\{H_{g_k}\}$ such that

$$h_{f_k} = (h_{f_k})^* \circ H_{f_k} = f_k^* \circ H_{f_k} \leqslant f^* \circ H_{f_k} \sim f^* \quad \text{for } k \in \mathbb{N},$$

and similarly for g_k. Therefore,

$$\begin{aligned}
\left\| \int_s^1 \frac{h_k(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{\overline{Y}(0,1)} &\leqslant \left\| \int_s^1 \frac{h_{f_k}(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{\overline{Y}(0,1)} + \left\| \int_s^1 \frac{h_{g_k}(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{\overline{Y}(0,1)} \\
&\leqslant \left\| \int_s^1 \frac{f^* \circ H_{f_k}(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{\overline{Y}(0,1)} + \left\| \int_s^1 \frac{g^* \circ H_{g_k}(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{\overline{Y}(0,1)} \\
&\leqslant \|f\|_{\overline{X}(0,1)} + \|g\|_{\overline{X}(0,1)},
\end{aligned} \tag{4.20}$$

for $k \in \mathbb{N}$. Since, by (4.13),

$$\|f+g\|_{\overline{X}(0,1)} \leqslant \big\| |f| + |g| \big\|_{\overline{X}(0,1)},$$

we get from (4.19) and (4.20) that

$$\|f+g\|_{\overline{X}(0,1)} \leqslant \|f\|_{\overline{X}(0,1)} + \|g\|_{\overline{X}(0,1)}.$$

The triangle inequality for $\|\cdot\|_{\overline{X}(0,1)}$ is thus established.

We now pass to the proof of properties (P1)–(P5). The lattice property (P1) is a cosequence of (4.13). As for property (P2), suppose that $\{f_k\}$ is a sequence in $\mathcal{M}_+(0,1)$ such that $f_k \nearrow f$ a.e. in $(0,1)$. By (4.13), we have that $\|f_k\|_{\overline{X}(0,1)} \leqslant \|f_{k+1}\|_{\overline{X}(0,1)}$ for $k \in \mathbb{N}$. Furthermore, if h is any function such that $h \sim f$, then $h = f^* \circ H$ for some measure-preserving transformation H. Consequently, we have that $f_k \sim f_k^* \circ H \nearrow f^* \circ H = h \sim f$ for $k \in \mathbb{N}$, whence $\|f_k\|_{\overline{X}(0,1)} \nearrow \|f\|_{\overline{X}(0,1)}$. To prove (P3), note that, by (4.2),

$$\|1\|_{\overline{X}(0,1)} \leqslant C \left\| \sqrt{1+\log\frac{1}{s}} \right\|_{\overline{Y}(0,1)} \leqslant C' \left\| \sqrt{1+\log\frac{1}{s}} \right\|_{\exp L^2(0,1)} < \infty,$$

for some absolute constant C and for some constant $C' = C'(Y)$. Finally, by (4.2) again,

$$\|f\|_{\overline{X}(0,1)} \geqslant \left\| \int_s^1 \frac{|f(r)|}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{\overline{Y}(0,1)} \geqslant C \left\| \int_s^1 \frac{|f(r)|}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{L(\log L)^{\frac{1}{2}}(0,1)} \geqslant C' \|f\|_{L^1(0,1)},$$

for some constants $C = C(Y)$ and $C' = C'(Y)$ and for every $f \in \overline{X}(0,1)$. This establishes (P4). Since (P5) is obvious, the proof is complete. $\square$

Theorem 4.3 will follow from Lemma 4.2, via the next result.

Lemma 4.5. *Let $Y(0,1)$ and $Z(0,1)$ be r.i. spaces. Assume that the operator T satisfies*

$$T : Y'(0,1) \to Z'(0,1). \tag{4.21}$$

Then there exists a constant $C = C(Y, Z)$ such that

$$\left\| \int_s^1 \frac{f(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{Y(0,1)} \leqslant C \left\| \int_s^1 \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{Z(0,1)} \tag{4.22}$$

for every $f \in \mathcal{M}_+(0,1)$.

Proof. The conclusion is a consequence of the following chain, which holds for every $f \in \mathcal{M}_+(0,1)$:

$$\begin{aligned}
\left\| \int_s^1 \frac{f(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{Y(0,1)} &= \sup_{\|g\|_{Y'(0,1)}\leqslant 1} \int_0^1 g^*(s) \int_s^1 \frac{f(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr\,ds \\
&= \sup_{\|g\|_{Y'(0,1)}\leqslant 1} \int_0^1 \frac{f(r)}{r\sqrt{1+\log\frac{1}{r}}} \int_0^r g^*(s)\,ds\,dr \\
&= \sup_{\|g\|_{Y'(0,1)}\leqslant 1} \int_0^1 f(r) \frac{g^{**}(r)}{\sqrt{1+\log\frac{1}{r}}}\,dr \\
&\leqslant \sup_{\|g\|_{Y'(0,1)}\leqslant 1} \int_0^1 \frac{f(r)}{\sqrt{1+\log\frac{1}{r}}} (Tg)^{**}(r)\,dr \\
&\leqslant C \sup_{\|g\|_{Y'(0,1)}\leqslant 1} \int_0^1 \frac{f^*(r)}{\sqrt{1+\log\frac{1}{r}}} (Tg)^{**}(r)\,dr \\
&\leqslant C' \sup_{\|Tg\|_{Z'(0,1)}\leqslant 1} \int_0^1 \frac{f^*(r)}{\sqrt{1+\log\frac{1}{r}}} (Tg)^{**}(r)\,dr \\
&\leqslant C' \sup_{\|h\|_{Z'(0,1)}\leqslant 1} \int_0^1 h^*(r) \int_r^1 \frac{f^*(s)}{s\sqrt{1+\log\frac{1}{s}}}\,ds\,dr \\
&= C' \left\| \int_r^1 \frac{f^*(s)}{s\sqrt{1+\log\frac{1}{s}}}\,ds \right\|_{Z(0,1)},
\end{aligned}$$

for some absolute constant C and for some constant $C' = C'(Y, Z)$. Note that the first inequality in this chain holds since, trivially, $g^* \leqslant Tg$ for every $g \in \mathcal{M}(0,1)$. The second inequality follows via the Hardy–Littlewood inequality (2.3), since, for any function $g \in \mathcal{M}(0,1)$,

$$\frac{(Tg)^{**}(r)}{\sqrt{1+\log\frac{1}{r}}} \approx \frac{\int_0^r \sqrt{1+\log\frac{1}{s}}\,\sup_{s\leqslant\rho\leqslant 1}\frac{g^*(\rho)}{\sqrt{1+\log\frac{1}{\rho}}}\,ds}{\int_0^r \sqrt{1+\log\frac{1}{s}}\,ds} \quad \text{for } r \in (0,1),$$

with absolute equivalence constants, and the latter is an integral mean of a non-increasing function $\sup_{s\leqslant\rho\leqslant 1}\frac{g^*(\rho)}{\sqrt{1+\log\frac{1}{\rho}}}$ with respect to the measure $\sqrt{1+\log\frac{1}{s}}\,ds$ over $(0,r)$, whence it is itself non-increasing in r (actually, this is the key reason for employing the operator T). The third inequality follows from (4.21). $\square$

Proof of Theorem 4.3. We begin by showing that assumption (4.6) implies the second embedding in (4.2). By property (P3) for $\overline{Y}'(0,1)$, the constant function $f(s) = 1$ belongs to $\overline{Y}'(0,1)$. By (4.6), $Tf(s) = \sqrt{1+\log\frac{1}{s}} \in \overline{Y}'(0,1)$. This membership is equivalent to the embedding $\exp L^2(\mathbb{R}^n, \gamma_n) \to Y'(\mathbb{R}^n, \gamma_n)$, and the latter is in turn equivalent to the second embedding in (4.2).

As far as equivalence (4.7) is concerned, one trivially has

$$\left\| \int_s^1 \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{\overline{Y}(0,1)} \leqslant \|f\|_{\overline{X}(0,1)}$$

for any $f \in \mathcal{M}(0,1)$. Conversely, by (4.6) and Lemma 4.5 applied to the case when $Y(0,1) = \overline{Y}(0,1) = \overline{Z}(0,1) = Z(0,1)$, we obtain that

$$\|f\|_{\overline{X}(0,1)} \leqslant C \left\| \int_s^1 \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{\overline{Y}(0,1)}$$

for some constant $C = C(X, Y)$ and for every $f \in \mathcal{M}(0,1)$. The proof is complete. $\square$

In the proof of Proposition 4.4, and in other proofs below, we shall make use of the following characterization of a weighted Hardy inequality established in [30] and [29, Section 1.3.1, Theorem 2].

Proposition 4.6. *Let $1 \leqslant p \leqslant \infty$ and let v and $\omega \in \mathcal{M}_+(0,1)$.*

(i) *There exists a constant C such that*

$$\left\| \omega(s) \int_0^s f(r)\,dr \right\|_{L^p(0,1)} \leqslant C \|vf\|_{L^p(0,1)} \tag{4.23}$$

for every $f \in \mathcal{M}_+(0,1)$ *if and only if*

$$\sup_{0<s<1} \|\omega\chi_{(s,1)}\|_{L^p(0,1)} \left\| \frac{\chi_{(0,s)}}{\nu} \right\|_{L^{p'}(0,1)} < \infty. \tag{4.24}$$

Moreover, the best constant C *in* (4.23) *is equivalent to the left-hand side of* (4.24)*, up to constants depending on* p.

(ii) *There exists a constant* C *such that*

$$\left\| \omega(s) \int_s^1 f(r)\,dr \right\|_{L^p(0,1)} \leqslant C\|\nu f\|_{L^p(0,1)} \tag{4.25}$$

for every $f \in \mathcal{M}_+(0,1)$ *if and only if*

$$\sup_{0<s<1} \|\omega\chi_{(0,s)}\|_{L^p(0,1)} \left\| \frac{\chi_{(s,1)}}{\nu} \right\|_{L^{p'}(0,1)} < \infty. \tag{4.26}$$

Moreover, the best constant C *in* (4.25) *is equivalent to the left-hand side of* (4.26)*, up to constants depending on* p.

Expressions having the form

$$\sup_{s\leqslant r\leqslant 1} \frac{f^*(r)}{\sqrt{1+\log\frac{1}{r}}} \quad \text{and} \quad \sup_{s\leqslant r\leqslant 1} \frac{f^{**}(r)}{\sqrt{1+\log\frac{1}{r}}},$$

where $f \in \mathcal{M}(0,1)$, will come into play as well. In particular, the following proposition, a special case of a more general result in [24, Theorems 3.2 and 3.5], will be needed.

Proposition 4.7. *Let* $p \in [1,\infty)$*, and let* $\nu, \omega \in \mathcal{M}_+(0,1)$.

(i) *There exists a constant* C *such that*

$$\int_0^1 \left(\sup_{s\leqslant r\leqslant 1} \frac{f^*(r)}{\sqrt{1+\log\frac{1}{r}}} \right)^p \omega(s)\,ds \leqslant C \int_0^1 f^*(s)^p \nu(s)\,ds \tag{4.27}$$

for every $f \in \mathcal{M}(0,1)$*, if and only if*

$$\sup_{0<s<1} \frac{\int_0^s \omega(r)\,dr}{(1+\log\frac{1}{s})^{\frac{p}{2}} \int_0^s \nu(r)\,dr} < \infty. \tag{4.28}$$

Moreover, the best constant C *in* (4.27) *is equivalent to the left-hand side of* (4.28)*, up to constants depending on* p.

(ii) *There exists a constant C such that*

$$\int_0^1 \left(\sup_{s \leqslant r \leqslant 1} \frac{f^{**}(r)}{\sqrt{1+\log\frac{1}{r}}} \right)^p \omega(s)\, ds \leqslant C \int_0^1 f^*(s)^p \nu(s)\, ds \tag{4.29}$$

for every $f \in \mathcal{M}(0,1)$, if and only if either $p = 1$ and

$$\sup_{0<s<1} \frac{s \int_s^1 \frac{\omega(r)}{r\sqrt{1+\log\frac{1}{r}}}\, dr}{\int_0^s \nu(r)\, dr} < \infty, \tag{4.30}$$

or $1 < p < \infty$, (4.28) *holds, and*

$$\sup_{0<s<1} \left(\int_s^1 \left(\frac{1}{r\sqrt{1+\log\frac{1}{r}}} \right)^p \omega(r)\, dr \right)^{\frac{1}{p}} \left(\int_0^s \left(\frac{r}{\int_0^r \nu(\rho)\, d\rho} \right)^{p'} \nu(r)\, dr \right)^{\frac{1}{p'}} < \infty. \tag{4.31}$$

Moreover, the best constant C in (4.29) *is equivalent to the left-hand side of* (4.30) *if $p = 1$, and to the sum of the left-hand sides of* (4.28) *and* (4.31) *if $1 < p < \infty$, up to constants depending only on p.*

We next establish some results concerning the operator T to be used in our proofs.

Lemma 4.8. *There exists an absolute constant C such that*

$$(Tf)^{**}(s) \leqslant CT(f^{**})(s) \quad \textit{for } s \in (0,1), \tag{4.32}$$

for every $f \in \mathcal{M}(0,1)$.

Proof. Let $f \in \mathcal{M}(0,1)$. We will show that

$$(Tf)^{**}(s) \leqslant C\big(Tf(s) + f^{**}(s)\big) \quad \text{for } s \in (0,1), \tag{4.33}$$

for some absolute constant C, whence (4.32) obviously follows. To verify (4.33), note that

$$\begin{aligned}
\int_0^s (Tf)^*(r)\, dr &= \int_0^s \sqrt{1+\log\frac{1}{r}} \sup_{r \leqslant \rho \leqslant 1} \frac{f^*(\rho)}{\sqrt{1+\log\frac{1}{\rho}}}\, dr \\
&\leqslant \int_0^s \sqrt{1+\log\frac{1}{r}} \sup_{r \leqslant \rho \leqslant s} \frac{f^*(\rho)}{\sqrt{1+\log\frac{1}{\rho}}}\, dr \\
&\quad + \int_0^s \sqrt{1+\log\frac{1}{r}} \sup_{s \leqslant \rho \leqslant 1} \frac{f^*(\rho)}{\sqrt{1+\log\frac{1}{\rho}}}\, dr \quad \text{for } s \in (0,1).
\end{aligned}$$

We have that

$$\int_0^s \sqrt{1+\log\frac{1}{r}} \sup_{s\leqslant\rho\leqslant 1} \frac{f^*(\rho)}{\sqrt{1+\log\frac{1}{\rho}}}\,dr \leqslant Cs\sqrt{1+\log\frac{1}{s}} \sup_{s\leqslant\rho\leqslant 1} \frac{f^*(\rho)}{\sqrt{1+\log\frac{1}{\rho}}}$$
$$= Cs(Tf)(s) \quad \text{for } s\in(0,1),$$

for some absolute constant C. On the other hand,

$$\begin{aligned}
\int_0^s \sqrt{1+\log\frac{1}{r}} \sup_{r\leqslant\rho\leqslant s} \frac{f^*(\rho)}{\sqrt{1+\log\frac{1}{\rho}}}\,dr &\leqslant C\int_0^s \sqrt{1+\log\frac{1}{r}} \sup_{r\leqslant\rho\leqslant s} f^*(\rho)\int_{\frac{\rho}{2}}^{\rho} \frac{d\tau}{\tau\sqrt{1+\log\frac{1}{\tau}}}\,dr\\
&\leqslant C\int_0^s \sqrt{1+\log\frac{1}{r}} \sup_{r\leqslant\rho\leqslant s} \int_{\frac{\rho}{2}}^{\rho} \frac{f^*(\tau)}{\tau\sqrt{1+\log\frac{1}{\tau}}}\,d\tau\,dr\\
&\leqslant C\int_0^s \sqrt{1+\log\frac{1}{r}} \int_{\frac{r}{2}}^{s} \frac{f^*(\tau)}{\tau\sqrt{1+\log\frac{1}{\tau}}}\,d\tau\,dr\\
&\leqslant C\int_0^s \frac{f^*(\tau)}{\tau\sqrt{1+\log\frac{1}{\tau}}} \int_0^{2\tau} \sqrt{1+\log\frac{1}{r}}\,dr\,d\tau\\
&\leqslant C'\int_0^s f^*(\tau)\,d\tau = C'sf^{**}(s) \quad \text{for } s\in(0,1),
\end{aligned}$$

for some absolute constants C and C'. Combining these estimates yields (4.33). □

Corollary 4.9. *Let $p\in(0,\infty)$ and let $\omega\in\mathcal{M}_+(0,1)$. If the operator T is bounded on $\Lambda^p(\omega)(0,1)$, then it is bounded also on $\Gamma^p(\omega)(0,1)$.*

Proof. Let $f\in\mathcal{M}(0,1)$. By (4.32),

$$\begin{aligned}
\|Tf\|_{\Gamma^p(\omega)(0,1)} &= \left(\int_0^1 (Tf)^{**}(s)^p\omega(s)\,ds\right)^{\frac{1}{p}} \leqslant C\left(\int_0^1 T(f^{**})(s)^p\omega(s)\,ds\right)^{\frac{1}{p}}\\
&\leqslant C'\left(\int_0^1 f^{**}(s)^p\omega(s)\,ds\right)^{\frac{1}{p}} = C'\|f\|_{\Gamma^p(\omega)(0,1)}
\end{aligned}$$

for some absolute constant C and for some constant $C'=C'(p,\omega)$. □

Lemma 4.10. *Let p, q and α be such that one of the conditions in* (2.13) *is satisfied. Then the operator T is bounded on the Lorentz–Zygmund space $L^{p,q;\alpha}(0,1)$ if and only if one of the following conditions holds*:

$$\begin{cases} p=q=1, \quad \alpha \geqslant 0; \\ 1<p<\infty, \quad 1 \leqslant q \leqslant \infty, \quad \alpha \in \mathbb{R}; \\ p=\infty, \quad 1 \leqslant q < \infty, \quad \left(\alpha+\frac{1}{2}\right)q<-1; \\ p=q=\infty, \quad \alpha \leqslant -\frac{1}{2}. \end{cases} \tag{4.34}$$

Moreover, if one of the conditions in (4.34) *holds, then T is bounded also on $L^{(p,q;\alpha)}(0,1)$.*

Proof, sketched. The first assertion follows from Proposition 4.7. The second assertion is a straightforward consequence of Corollary 4.9. □

Remark 4.11. Note that, by Lemma 4.10, the operator T is bounded on every Lebesgue space $L^p(0,1)$, with $p \in [1,\infty)$, on every Zygmund space $L^p(\log L)^\alpha(0,1)$, where either $p \in (1,\infty)$ and $\alpha \in \mathbb{R}$, or $p=1$ and $\alpha \geqslant 0$, and on every exponential space $\exp L^\beta(0,1)$, with $\beta \leqslant 2$. Instead, it is neither bounded on $L^\infty(0,1)$ nor on any exponential space $\exp L^\beta(0,1)$, with $\beta > 2$.

We are now ready to prove Proposition 4.4.

Proof of Proposition 4.4. Throughout the proof, f denotes any function in $\mathcal{M}(0,1)$.

(i) From Theorem 4.1 and the weighted Hardy inequality (see Proposition 4.6), we have that the optimal range $Y(\mathbb{R}^n, \gamma_n)$ for $L^p(\mathbb{R}^n, \gamma_n)$ fulfils

$$\|f\|_{\overline{Y}'(0,1)} = \left\| \frac{f^{**}(s)}{\sqrt{1+\log\frac{1}{s}}} \right\|_{L^{p'}(0,1)} \approx \left\| \frac{f^{*}(s)}{\sqrt{1+\log\frac{1}{s}}} \right\|_{L^{p'}(0,1)},$$

with equivalence constants depending on p. Since the function $\frac{1}{\sqrt{1+\log\frac{1}{s}}}$ is increasing on $(0,1)$, an application of [21, Theorem 2.7] tells us that

$$\|f\|_{\overline{Y}(0,1)} \approx \|f\|_{L^p(\log L)^{\frac{p}{2}}(0,1)},$$

with equivalence constants depending on p. This proves the optimality of $L^p(\log L)^{\frac{p}{2}}(\mathbb{R}^n, \gamma_n)$ as a range for $L^p(\mathbb{R}^n, \gamma_n)$.

As for the optimality of the domain, let $Y(\mathbb{R}^n, \gamma_n) = L^p(\log L)^{\frac{p}{2}}(\mathbb{R}^n, \gamma_n)$. Assumption (4.5) is clearly satisfied. Moreover, $Y'(\mathbb{R}^n, \gamma_n) = L^{p'}(\log L)^{\frac{p'}{2}}(\mathbb{R}^n, \gamma_n)$ when $p>1$ and $Y'(\mathbb{R}^n, \gamma_n) = \exp L^2(\mathbb{R}^n, \gamma_n)$ when $p=1$. Thus, in any case, T is bounded on $\overline{Y}'(0,1)$, by Lemma 4.10, and hence assumption (4.6) is also fulfilled. By Theorem 4.3, the norm in the optimal domain $X(\mathbb{R}^n, \gamma_n)$ satisfies

$$\|f\|_{\overline{X}(0,1)} \approx \left\| \int_s^1 \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{L^p(\log L)^{\frac{p}{2}}(0,1)} \approx \left\| \sqrt{1+\log\frac{1}{s}} \int_s^1 \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{L^p(0,1)},$$

with equivalence constants depending on p. By the weighted Hardy inequality (Proposition 4.6),

$$\left\| \sqrt{1+\log\frac{1}{s}} \int_s^1 \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{L^p(0,1)} \leqslant C\|f\|_{L^p(0,1)},$$

for some constant $C = C(p)$. Conversely,

$$\begin{aligned}
&\left\| \sqrt{1+\log\frac{1}{s}} \int_s^1 \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{L^p(0,1)} \\
&\quad \geqslant \left\| \chi_{(0,\frac{1}{2})}(s)\sqrt{1+\log\frac{1}{s}} \int_s^{2s} \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{L^p(0,1)} \\
&\quad \geqslant \left\| \chi_{(0,\frac{1}{2})}(s) f^*(2s)\sqrt{1+\log\frac{1}{s}} \int_s^{2s} \frac{dr}{r\sqrt{1+\log\frac{1}{r}}} \right\|_{L^p(0,1)} \\
&\quad \geqslant C\|f\|_{L^p(0,1)},
\end{aligned}$$

for some positive constant $C = C(p)$. Altogether, $L^p(\mathbb{R}^n, \gamma_n)$ is the optimal domain for $L^p(\log L)^{\frac{p}{2}}(\mathbb{R}^n, \gamma_n)$.

(ii) First, we show that the domain is optimal in (4.9). Let $Y(\mathbb{R}^n, \gamma_n) = \exp L^2(\mathbb{R}^n, \gamma_n)$. Then $Y'(\mathbb{R}^n, \gamma_n) = L(\log L)^{\frac{1}{2}}(\mathbb{R}^n, \gamma_n)$, and hence T is bounded on $\overline{Y}'(0,1)$, by Lemma 4.10. Thus, we can apply Theorem 4.3. One has that

$$\begin{aligned}
\left\| \int_s^1 \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{\exp L^2(0,1)} &\approx \sup_{0<s<1} \frac{1}{\sqrt{1+\log\frac{1}{r}}} \int_s^1 \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \\
&\leqslant \|f\|_{L^\infty(0,1)} \sup_{0<s<1} \frac{1}{\sqrt{1+\log\frac{1}{s}}} \int_s^1 \frac{dr}{r\sqrt{1+\log\frac{1}{r}}} \\
&\approx \|f\|_{L^\infty(0,1)},
\end{aligned}$$

with absolute equivalence constants. Conversely,

$$\left\| \int_s^1 \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{\exp L^2(0,1)} \approx \sup_{0<s<1} \frac{1}{\sqrt{1+\log\frac{1}{s}}} \int_s^1 \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr$$
$$\geqslant \lim_{s\to 0+} \frac{1}{\sqrt{1+\log\frac{1}{s}}} \int_s^1 \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr$$
$$= 2\lim_{s\to 0+} f^*(s) = 2\|f\|_{L^\infty(0,1)}.$$

By Theorem 4.3, this shows that $L^\infty(\mathbb{R}^n,\gamma_n)$ is the optimal domain for $\exp L^2(\mathbb{R}^n,\gamma_n)$.

Assume now that $Y(\mathbb{R}^n,\gamma_n)$ is the optimal range for $L^\infty(\mathbb{R}^n,\gamma_n)$. Then, by Theorem 4.1,

$$\|f\|_{\overline{Y}'(0,1)} = \left\| \frac{f^{**}(s)}{\sqrt{1+\log\frac{1}{s}}} \right\|_{L^1(0,1)} \approx \int_0^1 f^*(s)\sqrt{1+\log\frac{1}{s}}\,ds \approx \|f\|_{L(\log L)^{\frac{1}{2}}(0,1)},$$

with absolute equivalence constants. Since, by (2.14), $(\exp L^2)'(\mathbb{R}^n,\gamma_n) = L(\log L)^{\frac{1}{2}}(\mathbb{R}^n,\gamma_n)$, we deduce that $Y(\mathbb{R}^n,\gamma_n) = \exp L^2(\mathbb{R}^n,\gamma_n)$.

(iii) By Theorem 4.1, the optimal range $Y(\mathbb{R}^n,\gamma_n)$ for the domain $\exp L^\beta(\mathbb{R}^n,\gamma_n)$ satisfies the chain

$$\|f\|_{\overline{Y}'(0,1)} \approx \left\| \frac{f^{**}(s)}{\sqrt{1+\log\frac{1}{s}}} \right\|_{L(\log L)^{\frac{1}{\beta}}(0,1)} \approx \int_0^1 \left[\frac{f^{**}(\cdot)}{\sqrt{1+\log\frac{1}{(\cdot)}}} \right]^*(s) \left(1+\log\frac{1}{s}\right)^{\frac{1}{\beta}} ds$$
$$\leqslant \int_0^1 \sup_{s\leqslant r\leqslant 1} \frac{f^{**}(r)}{\sqrt{1+\log\frac{1}{r}}} \left(1+\log\frac{1}{s}\right)^{\frac{1}{\beta}} ds \leqslant C\int_0^1 f^*(s)\left(1+\log\frac{1}{s}\right)^{\frac{1}{\beta}+\frac{1}{2}} ds$$
$$\approx \|f\|_{L(\log L)^{\frac{2+\beta}{2\beta}}(0,1)},$$

with the equivalence constants and the constant C depending on β. Note that the last inequality follows from Proposition 4.7(ii). Conversely, by the Hardy–Littlewood inequality (2.3) and by Fubini's theorem,

$$\|f\|_{\overline{Y}'(0,1)} \approx \int_0^1 \left[\frac{f^{**}(\cdot)}{\sqrt{1+\log\frac{1}{(\cdot)}}} \right]^*(s) \left(1+\log\frac{1}{s}\right)^{\frac{1}{\beta}} ds \geqslant \int_0^1 \frac{f^{**}(s)}{\sqrt{1+\log\frac{1}{s}}} \left(1+\log\frac{1}{s}\right)^{\frac{1}{\beta}} ds$$
$$= \int_0^1 f^*(r) \int_r^1 \frac{1}{s}\left(1+\log\frac{1}{s}\right)^{\frac{1}{\beta}-\frac{1}{2}} ds\,dr \approx \|f\|_{L(\log L)^{\frac{2+\beta}{2\beta}}(0,1)},$$

with equivalence constants depending on β. Therefore, $Y'(\mathbb{R}^n,\gamma_n) = L(\log L)^{\frac{2+\beta}{2\beta}}(\mathbb{R}^n,\gamma_n)$, whence, by (2.14), $Y(\mathbb{R}^n,\gamma_n) = \exp L^{\frac{2\beta}{2+\beta}}(\mathbb{R}^n,\gamma_n)$.

In order to prove that the domain in (4.10) is optimal, set $Y(\mathbb{R}^n,\gamma_n)=\exp L^{\frac{2\beta}{2+\beta}}(\mathbb{R}^n,\gamma_n)$, and note that T is bounded on $\overline{Y}'(0,1)$. This follows via Proposition 4.7(i), which entails that

$$\|Tf\|_{L(\log L)^{\frac{2+\beta}{2\beta}}(0,1)} \approx \int_0^1 \sup_{s\leqslant r\leqslant 1} \frac{f^*(r)}{\sqrt{1+\log\frac{1}{r}}}\left(1+\log\frac{1}{s}\right)^{\frac{1}{\beta}+1} ds$$
$$\leqslant C\int_0^1 f^*(s)\left(1+\log\frac{1}{s}\right)^{\frac{2+\beta}{2\beta}} ds \approx \|f\|_{L(\log L)^{\frac{2+\beta}{2\beta}}(0,1)},$$

with the constant C and the equivalence constants depending on β. Therefore, Theorem 4.3 implies that the optimal domain $X(\mathbb{R}^n,\gamma_n)$ for $Y(\mathbb{R}^n,\gamma_n)$ fulfils

$$\|f\|_{\overline{X}(0,1)} \approx \left\|\int_s^1 \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr\right\|_{\exp L^{\frac{2\beta}{2+\beta}}(0,1)} \approx \sup_{0<s<1} \frac{1}{(1+\log\frac{1}{s})^{\frac{2+\beta}{2\beta}}}\int_s^1 \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr,$$

with equivalence constants depending on β. Since

$$\|f\|_{\exp L^{\beta}(0,1)} \approx \sup_{0<s<1}\frac{f^*(s)}{(1+\log\frac{1}{s})^{\frac{1}{\beta}}},$$

it suffices to show that

$$\sup_{0<s<1} \frac{1}{(1+\log\frac{1}{s})^{\frac{2+\beta}{2\beta}}}\int_s^1 \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \approx \sup_{0<s<1}\frac{f^*(s)}{(1+\log\frac{1}{s})^{\frac{1}{\beta}}}, \tag{4.35}$$

with equivalence constants depending on β. We have that

$$\sup_{0<s<1} \frac{1}{(1+\log\frac{1}{s})^{\frac{2+\beta}{2\beta}}}\int_s^1 \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr$$
$$\leqslant \sup_{0<s<1}\frac{f^*(s)}{(1+\log\frac{1}{s})^{\frac{1}{\beta}}}\sup_{0<s<1} \frac{1}{(1+\log\frac{1}{s})^{\frac{2+\beta}{2\beta}}}\int_s^1 \frac{(1+\log\frac{1}{r})^{\frac{1}{\beta}}}{r\sqrt{1+\log\frac{1}{r}}}\,dr$$
$$\leqslant C\sup_{0<s<1}\frac{f^*(s)}{(1+\log\frac{1}{s})^{\frac{1}{\beta}}},$$

for some constant $C=C(\beta)$. To prove the converse estimate, we first observe that there exists an absolute positive constant C such that

$$\int_{s^2}^{s} \frac{dr}{r\sqrt{1+\log\frac{1}{r}}} \geqslant C\sqrt{1+\log\frac{1}{s}} \quad \text{for } s\in(0,1/2]. \tag{4.36}$$

Next, let $s_0 \in (0, 1]$ be such that

$$\frac{f^*(s_0)}{(1+\log\frac{1}{s_0})^{\frac{1}{\beta}}} \geqslant \frac{1}{2} \sup_{0<s<1} \frac{f^*(s)}{(1+\log\frac{1}{s})^{\frac{1}{\beta}}}.$$

Clearly,

$$\sup_{0<s<1} \frac{1}{(1+\log\frac{1}{s})^{\frac{2\beta}{2+\beta}}} \int_s^1 \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \geqslant \sup_{0<s<s_0} \frac{f^*(s_0)}{(1+\log\frac{1}{s})^{\frac{2\beta}{2+\beta}}} \int_s^{s_0} \frac{dr}{r\sqrt{1+\log\frac{1}{r}}}.$$

We will be done if we show that there exists an absolute constant $\delta > 0$ such that the right-hand side of the last inequality is not smaller than

$$\frac{\delta f^*(s_0)}{(1+\log\frac{1}{s_0})^{\frac{1}{\beta}}}.$$

This is easily seen when $s_0 \in [\frac{1}{2}, 1]$: one has just to estimate the supremum from below by the value at, say, $s = \frac{1}{4}$. In the case when $s \in (0, \frac{1}{2}]$ it suffices to estimate the supremum by the value at $s = s_0^2$ and make use of (4.36). $\square$

5. Orlicz spaces

Here, we establish an optimal Gaussian Sobolev inequality for the Orlicz–Sobolev space $V^1L^A(\mathbb{R}^n, \gamma_n)$ associated with a Young function A.

We may assume, without loss of generality, that

$$\int_0 \frac{\widetilde{A}(t)}{t^2}\,dt < \infty. \tag{5.1}$$

Actually, A can be replaced, if necessary, by a Young function equivalent near infinity and fulfilling (5.1), without changing $L^A(\mathbb{R}^n, \gamma_n)$ (up to equivalent norms).

Let $E : (0, \infty) \to [0, \infty)$ be the (non-decreasing) function obeying

$$E^{-1}(t) = \left\| \frac{1}{r\sqrt{1+\log_+\frac{1}{r}}} \right\|_{L^{\widetilde{A}}(\frac{1}{t},\infty)} \quad \text{for } t > 0, \tag{5.2}$$

where $\log_+ t = \max\{\log t, 0\}$. Note that the right-hand side of (5.2) is actually finite for $t > 0$, owing to (5.1). Define $A_G : [0, \infty) \to [0, \infty)$ by

$$A_G(t) = \int_0^t \frac{E(s)}{s}\,ds \quad \text{for } t > 0. \tag{5.3}$$

The main result of this section tells us that $L^{A_G}(\mathbb{R}^n,\gamma_n)$ is the optimal Orlicz space into which $V^1L^A(\mathbb{R}^n,\gamma_n)$ is continuously embedded.

Theorem 5.1. *Let A be a Young function* (*modified, if necessary, near* 0 *in such a way that* (5.1) *is satisfied*)*. Then, an absolute constant C exists such that*

$$\|u-u_{\gamma_n}\|_{L^{A_G}(\mathbb{R}^n,\gamma_n)} \leqslant C\|\nabla u\|_{L^A(\mathbb{R}^n,\gamma_n)} \tag{5.4}$$

for every $u\in V^1L^A(\mathbb{R}^n,\gamma_n)$. Moreover, $L^{A_G}(\mathbb{R}^n,\gamma_n)$ is the optimal Orlicz range space in (5.4).

A special Orlicz space $L^A(\mathbb{R}^n,\gamma_n)$ which is self-optimal in the Sobolev inequality (5.4) will be exhibited in Corollary 7.2, Section 7.

We split the proof of Theorem 5.1 in some lemmas. We begin by showing that A_G is actually a Young function.

Lemma 5.2. *Let A be a Young function fulfilling* (5.1)*. Then A_G is a Young function. Moreover,*

$$E(t/2)\leqslant A_G(t)\leqslant E(t) \quad \textit{for } t>0. \tag{5.5}$$

Proof. In order to prove that A_G is a Young function, it suffices to show that the function $\frac{E(t)}{t}$ is non-decreasing in $(0,\infty)$, or, equivalently, that the function

$$t\left\|\frac{1}{s\sqrt{1+\log_+\frac{1}{s}}}\right\|_{L^{\widetilde{A}}(t,\infty)}$$

is non-decreasing in $(0,\infty)$. We have that

$$\begin{aligned} t\left\|\frac{1}{s\sqrt{1+\log_+\frac{1}{s}}}\right\|_{L^{\widetilde{A}}(t,\infty)} &= \inf\left\{\lambda\geqslant 0:\ \int_t^\infty \widetilde{A}\left(\frac{t}{\lambda s\sqrt{1+\log_+\frac{1}{s}}}\right)ds\leqslant 1\right\} \\ &= \inf\left\{\lambda\geqslant 0:\ \int_1^\infty \widetilde{A}\left(\frac{1}{\lambda s\sqrt{1+\log_+\frac{1}{ts}}}\right)t\,ds\leqslant 1\right\} \quad \text{for } t>0. \end{aligned} \tag{5.6}$$

Since the function $\widetilde{A}\big(\frac{1}{\lambda s\sqrt{1+\log_+\frac{1}{ts}}}\big)$ is non-decreasing in t for each λ and s, the function $t\|\frac{1}{s\sqrt{1+\log_+\frac{1}{s}}}\|_{L^{\widetilde{A}}(t,\infty)}$ is non-decreasing as well.

As far as (5.5) is concerned, owing to the monotonicity of the function $\frac{E(t)}{t}$, one has that

$$E(t/2)\leqslant \int_{\frac{t}{2}}^t \frac{E(s)}{s}\,ds\leqslant A_G(t)=\int_0^t\frac{E(s)}{s}\,ds\leqslant E(t) \quad \text{for } t>0. \qquad \square$$

Define the operator R as

$$Rf(s)=\int_s^1 \frac{f(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \quad \text{for } s\in(0,1), \tag{5.7}$$

for $f\in\mathcal{M}_+(0,1)$.

Lemma 5.3. *Let A be as in Lemma 5.2. Then,*

$$\bigl|\{Rf\geqslant t\}\bigr|A_G(t/2)\leqslant\int_0^1 A\bigl(f(s)\bigr)\,ds \quad \textit{for } t\geqslant 0, \tag{5.8}$$

for every $f\in\mathcal{M}_+(0,1)$ such that

$$\int_0^1 A\bigl(f(s)\bigr)\,ds\leqslant 1. \tag{5.9}$$

Here, and in what follows, $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}$.

Proof. One has, by (2.11),

$$\begin{aligned}Rf(s)=\int_s^1 \frac{f(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr &\leqslant 2\|f\|_{L^A(0,1)}\left\|\frac{1}{r\sqrt{1+\log\frac{1}{r}}}\right\|_{L^{\widetilde{A}}(s,1)}\\ &\leqslant 2\|f\|_{L^A(0,1)}E^{-1}(1/s) \quad \text{for } s\in(0,1).\end{aligned} \tag{5.10}$$

Thus,

$$\bigl|\{Rf\geqslant t\}\bigr|\leqslant\bigl|\bigl\{s\colon\, 2\|f\|_{L^A(0,1)}E^{-1}(1/s)\geqslant t\bigr\}\bigr|=\frac{1}{E(\frac{t}{2\|f\|_{L^A(0,1)}})} \quad \text{for } t>0, \tag{5.11}$$

whence

$$\bigl|\{Rf\geqslant t\}\bigr|E\left(\frac{t}{2\|f\|_{L^A(0,1)}}\right)\leqslant 1 \quad \text{for } t>0. \tag{5.12}$$

Next, given any nonnegative number M, define $A_M:[0,\infty)\to[0,\infty)$ as

$$A_M(t)=\frac{A(t)}{M} \quad \text{for } t\geqslant 0,$$

and denote by E_M the function defined as in (5.2) with A replaced by A_M. We claim that, if $M \leqslant 1$, then

$$E_M(t) \geqslant \frac{1}{M} E(t) \quad \text{for } t > 0. \tag{5.13}$$

Indeed, it is easily seen that $\widetilde{A_M}(t) = \frac{1}{M}\widetilde{A}(Mt)$ for $t \geqslant 0$. Consequently,

$$\begin{aligned}
E_M^{-1}(1/t) &= \left\| \frac{1}{s\sqrt{1+\log_+ \frac{1}{s}}} \right\|_{L^{\widetilde{A_M}}(t,\infty)} \\
&= \inf\left\{ \lambda \geqslant 0\colon \int_t^\infty \frac{1}{M}\widetilde{A}\left(\frac{M}{\lambda s\sqrt{1+\log_+ \frac{1}{s}}}\right) ds \leqslant 1 \right\} \\
&= \inf\left\{ \lambda \geqslant 0\colon \int_{\frac{t}{M}}^\infty \widetilde{A}\left(\frac{1}{\lambda s\sqrt{1+\log_+ \frac{1}{Ms}}}\right) ds \leqslant 1 \right\} \\
&\leqslant \inf\left\{ \lambda \geqslant 0\colon \int_{\frac{t}{M}}^\infty \widetilde{A}\left(\frac{1}{\lambda s\sqrt{1+\log_+ \frac{1}{s}}}\right) ds \leqslant 1 \right\} \\
&= \left\| \frac{1}{s\sqrt{1+\log_+ \frac{1}{s}}} \right\|_{L^{\widetilde{A}}(\frac{t}{M},\infty)} \\
&= E^{-1}(M/t) \quad \text{for } t > 0,
\end{aligned} \tag{5.14}$$

whence (5.13) follows.

Now, choose

$$M = \int_0^1 A\big(f(s)\big)\, ds.$$

Assumption (5.9) entails that $M \leqslant 1$. The very definition of the Luxemburg norm yields that

$$\|f\|_{L^{A_M}(0,1)} \leqslant 1. \tag{5.15}$$

On applying (5.12) with A replaced by A_M and making use of (5.15) and (5.14), one gets that

$$\begin{aligned}
1 &\geqslant \big|\{Rf \geqslant t\}\big| E_M\left(\frac{t}{2\|f\|_{L^{A_M}(0,1)}}\right) \geqslant \big|\{Rf \geqslant t\}\big| E_M\left(\frac{t}{2}\right) \\
&\geqslant \big|\{Rf \geqslant t\}\big| \frac{E(\frac{t}{2})}{M} \quad \text{for } t > 0.
\end{aligned} \tag{5.16}$$

Combining (5.16) with (5.5) yields (5.8). □

Lemma 5.4. *Let A be a Young function as in Lemma* 5.2. *Then,*

$$\|Rf\|_{L^{A_G}(0,1)} \leqslant 8\|f\|_{L^A(0,1)} \tag{5.17}$$

for every $f \in L^A(0,1)$. Moreover, $L^{A_G}(0,1)$ is the optimal Orlicz space in (5.17)*, in the sense that if* (5.17) *holds with A_G replaced by any Young function B, then $L^{A_G}(0,1) \to L^B(0,1)$.*

Proof. Let f be any function in $\mathcal{M}_+(0,1)$ satisfying (5.9). Thus, in particular, $f \in L^1(0,1)$, and hence $Rf(s) < \infty$ for $s \in (0,1)$. One can easily restrict oneself to the case when $\lim_{s\to 0+} Rf(s) = \infty$. Let $\{s_k\}_{k\in\mathbb{Z}}$ be a sequence in $(0,1)$ such that

$$Rf(s_k) = 2^k \quad \text{for } k \in \mathbb{Z}. \tag{5.18}$$

Notice that s_k is non-increasing, since so is Rf. Set

$$f_k = f\chi_{[s_k,s_{k-1})} \quad \text{for } k \in \mathbb{Z}.$$

Thus, since

$$Rf(s) \leqslant Rf(s_k) = 2^k \quad \text{if } s \in (s_k, s_{k-1}),$$

$$\begin{aligned}\int_0^1 A_G\left(\frac{Rf(s)}{8}\right) ds &= \sum_{k\in\mathbb{Z}} \int_{s_k}^{s_{k-1}} A_G\left(\frac{Rf(s)}{8}\right) ds \\ &\leqslant \sum_{k\in\mathbb{Z}} \int_{s_k}^{s_{k-1}} A_G\left(\frac{2^k}{8}\right) ds = \sum_{k\in\mathbb{Z}} (s_{k-1} - s_k) A_G\left(2^{k-3}\right).\end{aligned} \tag{5.19}$$

Now, for every $k \in \mathbb{Z}$ and $s \in (s_k, s_{k-1})$, one has that

$$\begin{aligned}R(f_{k-1})(s) &\geqslant \int_{s_{k-1}}^1 f_{k-1}(r) \frac{1}{r\sqrt{1+\log\frac{1}{r}}} dr = \int_{s_{k-1}}^1 f(r)\chi_{[s_{k-1},s_{k-2})}(r) \frac{1}{r\sqrt{1+\log\frac{1}{r}}} dr \\ &= \int_{s_{k-1}}^{s_{k-2}} \frac{f(r)}{r\sqrt{1+\log\frac{1}{r}}} dr = Rf(s_{k-1}) - Rf(s_{k-2}) = 2^{k-2}.\end{aligned}$$

Consequently,

$$[s_k, s_{k-1}) \subset \left\{Rf_{k-1} \geqslant 2^{k-2}\right\} \quad \text{for } k \in \mathbb{Z}. \tag{5.20}$$

From inclusion (5.20) and Lemma 5.3 we obtain that

$$(s_{k-1}-s_k)A_G\bigl(2^{k-3}\bigr) \leqslant \bigl|\bigl\{Rf_{k-1} \geqslant 2^{k-2}\bigr\}\bigr| A_G\bigl(2^{k-3}\bigr) \leqslant \int_0^1 A\bigl(f_{k-1}(s)\bigr)\,ds \quad \text{for } k \in \mathbb{Z}. \tag{5.21}$$

Notice that such an application of Lemma 5.3 with f replaced by f_{k-1} is possible since, by (5.9),

$$\int_0^1 A\bigl(f_{k-1}(s)\bigr)\,ds \leqslant \int_0^1 A\bigl(f(s)\bigr)\,ds \leqslant 1 \quad \text{for } k \in \mathbb{Z}.$$

Combining (5.19) and (5.21) yields

$$\int_0^1 A_G\left(\frac{Rf(s)}{8}\right)ds \leqslant \sum_{k\in\mathbb{Z}} \int_0^1 A\bigl(f_{k-1}(s)\bigr)\,ds = \int_0^1 A\bigl(f(s)\bigr)\,ds \leqslant 1. \tag{5.22}$$

Thus, we have shown that

$$\int_0^1 A_G\left(\frac{Rf(s)}{8}\right)ds \leqslant 1$$

provided that (5.9) is fulfilled. Hence, (5.17) follows.

The proof of the sharpness of the space $L^{A_G}(0,1)$ amounts to showing that if B is any Young function such that

$$\|Rf\|_{L^B(0,1)} \leqslant C\|f\|_{L^A(0,1)} \tag{5.23}$$

for some constant C, and for every $f \in L^A(0,1)$, then constants C' and t_1 exist such that

$$B(t) \leqslant A_G(C't) \quad \text{for } t \geqslant t_1. \tag{5.24}$$

By a standard argument in the characterization of Hardy type inequalities (see e.g. [20]), one can show that a necessary condition for (5.23) to hold is the existence of a constant C such that

$$\|1\|_{L^B(0,t)} \left\| \frac{1}{r\sqrt{1+\log\frac{1}{r}}} \right\|_{L^{\widetilde{A}}(t,1)} \leqslant C \quad \text{for } t \in (0,1). \tag{5.25}$$

We have that

$$\|1\|_{L^B(0,t)} = \frac{1}{B^{-1}(\frac{1}{t})} \quad \text{for } t > 0. \tag{5.26}$$

Moreover, there obviously exists a constant $C = C(A)$ such that

$$\left\| \frac{1}{s\sqrt{1+\log_+ \frac{1}{s}}} \right\|_{L^{\widetilde{A}}(t,\infty)} \leqslant C \left\| \frac{1}{s\sqrt{1+\log \frac{1}{s}}} \right\|_{L^{\widetilde{A}}(t,1)} \quad \text{for } t \in (0,1/2). \tag{5.27}$$

From the first inequality in (5.5), (5.26) and (5.27) we get that

$$\frac{1}{2} A_G^{-1}(1/t) \leqslant E^{-1}(1/t) \leqslant C B^{-1}(1/t) \quad \text{for } t \in (0,1/2).$$

Hence inequality (5.24) follows. The proof is complete. □

Proof of Theorem 5.1. The conclusions are straightforward consequences of Theorem 3.2 and Lemma 5.4. □

6. Lorentz–Zygmund spaces

In this section we establish the following sharp Gaussian Sobolev inequality for Lorentz and, more generally, Lorentz–Zygmund spaces. Note that, according to (2.13), the conditions on the parameters p, q and α in the statement are required to ensure that $L^{p,q;\alpha}(\mathbb{R}^n,\gamma_n)$ is actually an r.i. space.

Theorem 6.1.

(i) *Assume that either $p = q = 1$ and $\alpha \geqslant 0$, or $p \in (1,\infty)$, $q \in [1,\infty]$ and $\alpha \in \mathbb{R}$. Then, there exists a constant $C = C(p,q,\alpha)$ such that*

$$\|u - u_{\gamma_n}\|_{L^{p,q;\alpha+\frac{1}{2}}(\mathbb{R}^n,\gamma_n)} \leqslant C \|\nabla u\|_{L^{p,q;\alpha}(\mathbb{R}^n,\gamma_n)} \tag{6.1}$$

for every $u \in V^1 L^{p,q;\alpha}(\mathbb{R}^n,\gamma_n)$. Moreover, $(L^{p,q;\alpha}(\mathbb{R}^n,\gamma_n), L^{p,q;\alpha+\frac{1}{2}}(\mathbb{R}^n,\gamma_n))$ is an optimal pair in (6.1).

(ii) *Assume that either $q \in [1,\infty)$ and $\alpha + \frac{1}{q} < 0$, or $q = \infty$ and $\alpha \leqslant 0$. Then, there exists a constant $C = C(q,\alpha)$ such that*

$$\|u - u_{\gamma_n}\|_{L^{\infty,q;\alpha-\frac{1}{2}}(\mathbb{R}^n,\gamma_n)} \leqslant C \|\nabla u\|_{L^{\infty,q;\alpha}(\mathbb{R}^n,\gamma_n)} \tag{6.2}$$

for every $u \in V^1 L^{\infty,q;\alpha}(\mathbb{R}^n,\gamma_n)$. Moreover, $(L^{\infty,q;\alpha}(\mathbb{R}^n,\gamma_n), L^{\infty,q;\alpha-\frac{1}{2}}(\mathbb{R}^n,\gamma_n))$ is an optimal pair in (6.2).

Observe that also the new embeddings of Theorem 6.1 exhibit the contrast between the gain of integrability from $|\nabla u|$ to u in Sobolev embeddings for spaces far from $V^1 L^\infty(\mathbb{R}^n,\gamma_n)$, and the loss of integrability for spaces near $V^1 L^\infty(\mathbb{R}^n,\gamma_n)$.

Proof. Throughout the proof, we denote by f an arbitrary function from $\mathcal{M}(0,1)$.

(i) We shall first prove the optimality of the range. Let $p = q = 1$ and $\alpha \geqslant 0$, and let $X(\mathbb{R}^n,\gamma_n) = L^{1,1;\alpha}(\mathbb{R}^n,\gamma_n)$. Then $X'(\mathbb{R}^n,\gamma_n) = L^{\infty,\infty;-\alpha}(\mathbb{R}^n,\gamma_n)$, by (2.14). Let $Y(\mathbb{R}^n,\gamma_n)$ be the optimal range for $X(\mathbb{R}^n,\gamma_n)$ in (1.3). Then, by Theorem 4.1,

$$\|f\|_{\overline{Y}'(0,1)} \approx \left\| \frac{f^{**}(s)}{\sqrt{1+\log\frac{1}{s}}} \right\|_{L^{\infty,\infty;-\alpha}(0,1)} \leqslant \sup_{0<s<1} \left(1+\log\frac{1}{s}\right)^{-\alpha} \sup_{s<r<1} \frac{f^{**}(r)}{\sqrt{1+\log\frac{1}{r}}}$$

$$= \sup_{0<r<1} \left(\frac{f^{**}(r)}{\sqrt{1+\log\frac{1}{r}}} \sup_{0<s<r} \left(1+\log\frac{1}{s}\right)^{-\alpha} \right) = \sup_{0<r<1} f^{**}(r)\left(1+\log\frac{1}{r}\right)^{-\alpha-\frac{1}{2}}$$

$$\leqslant C\|f\|_{L^{\infty,\infty;-\alpha-\frac{1}{2}}(0,1)},$$

with equivalence constants and C depending on α. Note that the last inequality holds by Proposition 4.6.

Conversely,

$$\|f\|_{\overline{Y}'(0,1)} \approx \left\| \frac{f^{**}(s)}{\sqrt{1+\log\frac{1}{s}}} \right\|_{L^{\infty,\infty;-\alpha}(0,1)}$$

$$\geqslant C \sup_{0<s<1} \left(1+\log\frac{1}{s}\right)^{-\alpha} \frac{1}{s} \int_0^s \left(\frac{f^{**}(\cdot)}{\sqrt{1+\log\frac{1}{(\cdot)}}} \right)^*(r)\,dr$$

$$\geqslant C \sup_{0<s<1} \left(1+\log\frac{1}{s}\right)^{-\alpha} \frac{1}{s} \int_0^s \frac{f^{**}(r)}{\sqrt{1+\log\frac{1}{r}}}\,dr$$

$$\geqslant C \sup_{0<s<1} f^{**}(s)\left(1+\log\frac{1}{s}\right)^{-\alpha} \frac{1}{s} \int_0^s \frac{dr}{\sqrt{1+\log\frac{1}{r}}}$$

$$\geqslant C' \sup_{0<s<1} f^{**}(s)\left(1+\log\frac{1}{s}\right)^{-\alpha-\frac{1}{2}} \geqslant C'\|f\|_{L^{\infty,\infty;-\alpha-\frac{1}{2}}(0,1)},$$

for some absolute constants C and C'. Here, the first inequality is a consequence of Proposition 4.6, and the second one of the Hardy–Littlewood inequality (2.3). Altogether, $Y'(\mathbb{R}^n,\gamma_n) = L^{\infty,\infty;-\alpha-\frac{1}{2}}(\mathbb{R}^n,\gamma_n)$, or, equivalently, $Y(\mathbb{R}^n,\gamma_n) = L(\log L)^{\alpha+\frac{1}{2}}(\mathbb{R}^n,\gamma_n)$.

Now assume that $1<p<\infty$, $1\leqslant q\leqslant\infty$ and $\alpha\in\mathbb{R}$, and let $X(\mathbb{R}^n,\gamma_n) = L^{p,q;\alpha}(\mathbb{R}^n,\gamma_n)$. Then $X'(\mathbb{R}^n,\gamma_n) = L^{p',q';-\alpha}(\mathbb{R}^n,\gamma_n)$. Let $Y(\mathbb{R}^n,\gamma_n)$ be the optimal range for $X(\mathbb{R}^n,\gamma_n)$ in (1.3). Then, by Theorem 4.1 and Proposition 4.7(i),

$$\|f\|_{\overline{Y}'(0,1)} \approx \left\| \frac{f^{**}(s)}{\sqrt{1+\log\frac{1}{s}}} \right\|_{L^{p',q';-\alpha}(0,1)} \leqslant \left\| s^{\frac{1}{p'}-\frac{1}{q'}} \left(1+\log\frac{1}{s}\right)^{-\alpha} \sup_{s\leqslant r\leqslant 1} \frac{f^{**}(r)}{\sqrt{1+\log\frac{1}{r}}} \right\|_{L^{q'}(0,1)}$$

$$\leqslant C \left\| s^{\frac{1}{p'}-\frac{1}{q'}} \left(1+\log\frac{1}{s}\right)^{-\alpha-\frac{1}{2}} f^{**}(s) \right\|_{L^{q'}(0,1)},$$

with equivalence constants and C depending on p, q and α. Conversely,

$$\|f\|_{\overline{Y}'(0,1)} \approx \left\| \frac{f^{**}(s)}{\sqrt{1+\log\frac{1}{s}}} \right\|_{L^{p',q';-\alpha}(0,1)} = \left\| s^{\frac{1}{p'}-\frac{1}{q'}} \left(1+\log\frac{1}{s}\right)^{-\alpha} \left(\frac{f^{**}(\cdot)}{\sqrt{1+\log\frac{1}{(\cdot)}}} \right)^{*}(s) \right\|_{L^{q'}(0,1)}$$

$$= \left\| \sqrt{1+\log\frac{1}{s}} \left(\frac{f^{**}(\cdot)}{\sqrt{1+\log\frac{1}{(\cdot)}}} \right)^{*}(s) \right\|_{L^{p',q',-\alpha-\frac{1}{2}}(0,1)},$$

with equivalence constants depending on p, q and α. Notice that the last equality holds since the function $\sqrt{1+\log\frac{1}{s}}$ is non-increasing in $(0,1)$. By the Hardy–Littlewood inequality (2.3),

$$\int_0^s \sqrt{1+\log\frac{1}{r}} \left(\frac{f^{**}(\cdot)}{\sqrt{1+\log\frac{1}{(\cdot)}}} \right)^{*}(r)\,dr \geqslant \int_0^s \sqrt{1+\log\frac{1}{r}}\, \frac{f^{**}(r)}{\sqrt{1+\log\frac{1}{r}}}\,dr$$

$$= \int_0^s f^{**}(r)\,dr \geqslant \int_0^s f^{*}(r)\,dr \quad \text{for } s \in (0,1).$$

Thus, by (2.2),

$$\left\| \sqrt{1+\log\frac{1}{s}} \left(\frac{f^{**}(\cdot)}{\sqrt{1+\log\frac{1}{(\cdot)}}} \right)^{*}(s) \right\|_{L^{p',q',-\alpha-\frac{1}{2}}(0,1)} \geqslant \|f\|_{L^{p',q';-\alpha-\frac{1}{2}}(0,1)}.$$

In conclusion, we have shown that $Y'(\mathbb{R}^n,\gamma_n) = L^{p',q';-\alpha-\frac{1}{2}}(\mathbb{R}^n,\gamma_n)$, whence $Y(\mathbb{R}^n,\gamma_n) = L^{p,q;\alpha+\frac{1}{2}}(\mathbb{R}^n,\gamma_n)$.

Next, we shall prove the optimality of the domain. Let $Y(\mathbb{R}^n,\gamma_n) = L^{p,q;\alpha+\frac{1}{2}}(\mathbb{R}^n,\gamma_n)$, where either $p=q=1$ and $\alpha \geqslant 0$, or $1<p<\infty$, $1 \leqslant q \leqslant \infty$ and $\alpha \in \mathbb{R}$. Let $X(\mathbb{R}^n,\gamma_n)$ be the optimal domain for $Y(\mathbb{R}^n,\gamma_n)$. Assumption (4.5) is clearly satisfied. Moreover, by Lemma 4.10, the operator T is bounded on $\overline{Y}'(0,1)$, and hence (4.6) is fulfilled as well.

Assume first that $p=q=1$ and $\alpha \geqslant 0$. Then, by Theorem 4.3,

$$\|f\|_{\overline{X}(0,1)} \approx \left\| \int_s^1 \frac{f^{*}(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{L^{1,1;\alpha+\frac{1}{2}}(0,1)} = \int_0^1 \int_s^1 \frac{f^{*}(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \left(1+\log\frac{1}{s}\right)^{\alpha+\frac{1}{2}} ds$$

$$= \int_0^1 \frac{f^{*}(r)}{r\sqrt{1+\log\frac{1}{r}}} \int_0^r \left(1+\log\frac{1}{s}\right)^{\alpha+\frac{1}{2}} ds\,dr \approx \|f\|_{L^{1,1;\alpha}(0,1)},$$

with equivalence constants depending on α.

Next, assume that $1<p<\infty$, $1 \leqslant q \leqslant \infty$ and $\alpha \in \mathbb{R}$. Then, by Theorem 4.3 and by the weighted Hardy inequality (Proposition 4.6),

$$\|f\|_{\overline{X}(0,1)} \approx \left\| \int_s^1 \frac{f^{*}(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{L^{p,q;\alpha+\frac{1}{2}}(0,1)}$$

$$= \left\| s^{\frac{1}{p}-\frac{1}{q}} \left(1+\log\frac{1}{s}\right)^{\alpha+\frac{1}{2}} \int_s^1 \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{L^q(0,1)}$$

$$\leqslant C \left\| s^{\frac{1}{p}-\frac{1}{q}} \left(1+\log\frac{1}{s}\right)^{\alpha} f^*(s) \right\|_{L^q(0,1)} = C\|f\|_{L^{p,q;\alpha}(0,1)},$$

with equivalence constants and C depending on p, q and α.

Conversely,

$$\|f\|_{\overline{X}(0,1)} \geqslant C \left\| \chi_{(0,\frac{1}{2})}(s) s^{\frac{1}{p}-\frac{1}{q}} \left(1+\log\frac{1}{s}\right)^{\alpha+\frac{1}{2}} \int_s^{2s} \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{L^q(0,1)}$$

$$\geqslant C \left\| \chi_{(0,\frac{1}{2})}(s) s^{\frac{1}{p}-\frac{1}{q}} \left(1+\log\frac{1}{s}\right)^{\alpha+\frac{1}{2}} f^*(2s) \int_s^{2s} \frac{dr}{r\sqrt{1+\log\frac{1}{r}}} \right\|_{L^q(0,1)}$$

$$\approx \|f\|_{L^{p,q;\alpha}(0,1)},$$

with equivalence constants and C depending on p, q and α.

(ii) Note that the case when $p = q = \infty$ and $\alpha \leqslant 0$ has been established in Proposition 4.4, cases (ii) and (iii). We thus have only to deal with the remaining case when $p = \infty$, $1 \leqslant q < \infty$ and $\alpha + \frac{1}{q} < 0$.

Let us preliminarily observe that, by the Hardy–Littlewood inequality (2.3) and Fubini's theorem,

$$\left(\frac{f^{**}(\cdot)}{\sqrt{1+\log\frac{1}{(\cdot)}}}\right)^{**}(s) = \frac{1}{s}\int_0^s \left(\frac{f^{**}(\cdot)}{\sqrt{1+\log\frac{1}{(\cdot)}}}\right)^{*}(r)\,dr \geqslant \frac{1}{s}\int_0^s \frac{f^{**}(r)}{\sqrt{1+\log\frac{1}{r}}}\,dr$$

$$= \frac{1}{s}\int_0^s f^*(\rho)\int_\rho^s \frac{dr}{r\sqrt{1+\log\frac{1}{r}}}\,d\rho \quad \text{for } s \in (0,1). \tag{6.3}$$

At this stage we have to distinguish the cases where $1 < q < \infty$ and $q = 1$.

Assume first that $1 < q < \infty$. We begin by proving that $L^{\infty,q;\alpha-\frac{1}{2}}(\mathbb{R}^n,\gamma_n)$ is the optimal range for $L^{\infty,q;\alpha}(\mathbb{R}^n,\gamma_n)$. Let $X(\mathbb{R}^n,\gamma_n) = L^{\infty,q;\alpha}(\mathbb{R}^n,\gamma_n)$. Then, by (2.14), $X'(\mathbb{R}^n,\gamma_n) = L^{(1,q';-\alpha-1)}(\mathbb{R}^n,\gamma_n)$. By Theorem 4.1, the optimal range $Y(\mathbb{R}^n,\gamma_n)$ for $X(\mathbb{R}^n,\gamma_n)$ fulfils

$$\|f\|_{\overline{Y}'(0,1)} \approx \left\| \frac{f^{**}(s)}{\sqrt{1+\log\frac{1}{s}}} \right\|_{L^{(1,q';-\alpha-1)}(0,1)} \leqslant \left\| \sup_{s\leqslant r\leqslant 1} \frac{f^{**}(r)}{\sqrt{1+\log\frac{1}{r}}} \right\|_{L^{(1,q';-\alpha-1)}(0,1)}$$

$$= \left(\int_0^1 \left[\frac{1}{s}\int_0^s \sup_{r\leqslant\rho\leqslant 1} \frac{f^{**}(\rho)}{\sqrt{1+\log\frac{1}{\rho}}}\,dr \right]^{q'} s^{q'-1}\left(1+\log\frac{1}{s}\right)^{(-\alpha-1)q'} ds \right)^{\frac{1}{q'}},$$

with equivalence constants depending on q and α. Since

$$\sup_{r\leqslant\rho\leqslant 1}\frac{f^{**}(\rho)}{\sqrt{1+\log\frac{1}{\rho}}}=\max\left\{\sup_{r\leqslant\rho\leqslant s}\frac{f^{**}(\rho)}{\sqrt{1+\log\frac{1}{\rho}}},\sup_{s\leqslant\rho\leqslant 1}\frac{f^{**}(\rho)}{\sqrt{1+\log\frac{1}{\rho}}}\right\}\quad\text{if } 0<r\leqslant s<1,$$

we have that

$$\begin{aligned}\|f\|_{\overline{Y}'(0,1)}&\leqslant C\left(\int_0^1\left[\frac{1}{s}\int_0^s\sup_{r\leqslant\rho\leqslant s}\frac{f^{**}(\rho)}{\sqrt{1+\log\frac{1}{\rho}}}\,dr\right]^{q'}s^{q'-1}\left(1+\log\frac{1}{s}\right)^{(-\alpha-1)q'}ds\right)^{\frac{1}{q'}}\\&\quad+C\left(\int_0^1\left[\sup_{s\leqslant\rho\leqslant 1}\frac{f^{**}(\rho)}{\sqrt{1+\log\frac{1}{\rho}}}\right]^{q'}s^{q'-1}\left(1+\log\frac{1}{s}\right)^{(-\alpha-1)q'}ds\right)^{\frac{1}{q'}},\end{aligned}$$

for some constant $C=C(q,\alpha)$. Let us call I_1 and I_2 the first and the second addend, respectively, on the right-hand side of the last inequality. By Proposition 4.7(i), one has that

$$I_2\leqslant C\left(\int_0^1 f^{**}(s)^{q'}s^{q'-1}\left(1+\log\frac{1}{s}\right)^{(-\alpha-\frac{1}{2})q'}ds\right)^{\frac{1}{q'}},$$

for some constant $C=C(q,\alpha)$. As for I_1, note that, by Proposition 4.7(i),

$$\int_0^s\sup_{r\leqslant\rho\leqslant s}\frac{f^{**}(\rho)}{\sqrt{1+\log\frac{1}{\rho}}}\,dr\leqslant C\int_0^s\frac{f^{**}(r)}{\sqrt{1+\log\frac{1}{r}}}\,dr\quad\text{for } s\in(0,1),$$

for some absolute constant C. Thus,

$$\begin{aligned}I_1&\leqslant C\left(\int_0^1\left(\int_0^s\frac{f^{**}(r)}{\sqrt{1+\log\frac{1}{r}}}\,dr\right)^{q'}\frac{1}{s}\left(1+\log\frac{1}{s}\right)^{(-\alpha-1)q'}ds\right)^{\frac{1}{q'}}\\&\leqslant C'\left(\int_0^1 f^{**}(s)^{q'}s^{q'-1}\left(1+\log\frac{1}{s}\right)^{(-\alpha-\frac{1}{2})q'}ds\right)^{\frac{1}{q'}},\end{aligned}$$

for some constants $C=C(q,\infty)$ and $C'=C'(q,\alpha)$, as a consequence of the weighted Hardy inequality (Proposition 4.6). Thus, we have shown that

$$\|f\|_{\overline{Y}'(0,1)}\leqslant C\|f\|_{L^{(1,q';-\alpha-\frac{1}{2})}(0,1)}\tag{6.4}$$

for some constant $C=C(q,\alpha)$.

Now, we have to establish a lower bound. On defining $\omega : (0,1) \to [0,\infty)$ as

$$\omega(s) = \frac{1}{s}\left(1+\log\frac{1}{s}\right)^{(-\alpha-1)q'} \quad \text{for } s \in (0,1),$$

and making use of (6.3), we get that

$$\|f\|^{q'}_{Y'(0,1)} \geqslant C\int_0^1\left(\int_0^s f^*(r)\int_r^s \frac{d\rho}{\rho\sqrt{1+\log\frac{1}{\rho}}}\,dr\right)^{q'}\omega(s)\,ds,$$

for some constant $C = C(q,\alpha)$. Thus, we only need to show that there exists a positive constant $C = C(q,\alpha)$ such that

$$\int_0^1\left(\int_0^s f^*(r)\int_r^s \frac{d\rho}{\rho\sqrt{1+\log\frac{1}{\rho}}}\,dr\right)^{q'}\omega(s)\,ds \geqslant C\int_0^1\left(\int_0^s f^*(r)\,dr\sqrt{1+\log\frac{1}{s}}\right)^{q'}\omega(s)\,ds. \tag{6.5}$$

In order to prove (6.5), we use a discretization argument making use of (4.36). Define the sequence $\{s_k\}$ as

$$s_k = 2^{-2^k}, \quad \text{for } k \in \mathbb{N}\cup\{0\}, \tag{6.6}$$

so that $s_0 = \frac{1}{2}$, $\lim_{k\to\infty} s_k = 0$ and, for every $k \in \mathbb{N}\cup\{0\}$, $s_{k+1} = s_k^2$. Therefore, via (4.36) one can easily verify that

$$\begin{aligned}
&\int_0^1\left(\int_0^s f^*(r)\int_r^s \frac{d\rho}{\rho\sqrt{1+\log\frac{1}{\rho}}}\,dr\right)^{q'}\omega(s)\,ds \\
&\quad\geqslant \sum_{k=0}^{\infty}\int_{s_{k+1}}^{s_k}\left(\int_0^s f^*(r)\int_r^s \frac{d\rho}{\rho\sqrt{1+\log\frac{1}{\rho}}}\,dr\right)^{q'}\omega(s)\,ds \\
&\quad\geqslant \sum_{k=0}^{\infty}\int_{s_{k+1}}^{s_k}\left(\int_0^s f^*(r)\int_r^{s_{k+1}} \frac{d\rho}{\rho\sqrt{1+\log\frac{1}{\rho}}}\,dr\right)^{q'}\omega(s)\,ds \\
&\quad\geqslant \sum_{k=0}^{\infty}\int_{s_{k+1}}^{s_k}\left(\int_0^{s_{k+2}} f^*(r)\,dr\int_{s_{k+2}}^{s_{k+1}} \frac{d\rho}{\rho\sqrt{1+\log\frac{1}{\rho}}}\right)^{q'}\omega(s)\,ds \\
&\quad\geqslant C\sum_{k=0}^{\infty}\left(\int_0^{s_{k+2}} f^*(r)\,dr\sqrt{1+\log\frac{1}{s_{k+1}}}\right)^{q'}\int_{s_{k+1}}^{s_k}\omega(s)\,ds
\end{aligned}$$

$$\geqslant C' \sum_{k=0}^{\infty} \left(\int_0^{s_{k+2}} f^*(r)\,dr \sqrt{1+\log\frac{1}{s_{k+3}}} \right)^{q'} \int_{s_{k+3}}^{s_{k+2}} \omega(s)\,ds$$

$$\geqslant C'' \sum_{k=0}^{\infty} \int_{s_{k+3}}^{s_{k+2}} \left(\int_0^{s} f^*(r)\,dr \sqrt{1+\log\frac{1}{s}} \right)^{q'} \omega(s)\,ds$$

$$= C'' \int_0^{\frac{1}{16}} \left(\int_0^{s} f^*(r)\,dr \sqrt{1+\log\frac{1}{s}} \right)^{q'} \omega(s)\,ds$$

$$\geqslant C''' \int_0^{1} \left(\int_0^{s} f^*(r)\,dr \sqrt{1+\log\frac{1}{s}} \right)^{q'} \omega(s)\,ds,$$

for some constants C, C', C'', C''' depending on q and α. Inequality (6.5) is thus established, and hence also the inequality

$$\|f\|_{\overline{Y}'(0,1)} \geqslant C \|f\|_{L^{(1,q';-\alpha-\frac{1}{2})}(0,1)} \tag{6.7}$$

for some constant $C = C(q,\alpha)$. Combining (6.4) and (6.7) yields $\overline{Y}'(0,1) = L^{(1,q';-\alpha-\frac{1}{2})}(0,1)$, whence, by (2.14), $Y(\mathbb{R}^n,\gamma_n) = L^{\infty,q;\alpha-\frac{1}{2}}(\mathbb{R}^n,\gamma_n)$.

Now, let us show that $L^{\infty,q;\alpha}(\mathbb{R}^n,\gamma_n)$ is the optimal domain for $L^{\infty,q;\alpha-\frac{1}{2}}(\mathbb{R}^n,\gamma_n)$. To this purpose, set $Y(\mathbb{R}^n,\gamma_n) = L^{\infty,q;\alpha-\frac{1}{2}}(\mathbb{R}^n,\gamma_n)$. First, observe that, by (2.15), $\exp L^2(\mathbb{R}^n,\gamma_n) \to Y(\mathbb{R}^n,\gamma_n)$, since $\exp L^2(\mathbb{R}^n,\gamma_n) = L^{\infty,\infty;-\frac{1}{2}}(\mathbb{R}^n,\gamma_n)$. Second, by (2.14), $\overline{Y}'(0,1) = L^{(1,q';-\alpha-\frac{1}{2})}(0,1)$. By Proposition 4.7(i), T is bounded on $L^{1,q';-\alpha-\frac{1}{2}}(0,1)$, and hence, by Corollary 4.9, also on $L^{(1,q';-\alpha-\frac{1}{2})}(0,1)$. Assumptions (4.5) and (4.6) of Theorem 4.3 are thus fulfilled. According to this theorem, the norm of the optimal domain $X(\mathbb{R}^n,\gamma_n)$ for $Y(\mathbb{R}^n,\gamma_n)$ satisfies

$$\|f\|_{\overline{X}(0,1)} \approx \left\| \int_s^1 \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{L^{\infty,q;\alpha-\frac{1}{2}}(0,1)}$$

$$= \left(\int_0^1 \left(\int_s^1 \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right)^q \left(1+\log\frac{1}{s}\right)^{q(\alpha-\frac{1}{2})} \frac{ds}{s} \right)^{\frac{1}{q}},$$

with equivalence constants depending on q and α. We will be done if we show that

$$\int_0^1 \left(\int_s^1 \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right)^q \left(1+\log\frac{1}{s}\right)^{q(\alpha-\frac{1}{2})} \frac{ds}{s} \approx \int_0^1 f^*(s)^q \left(1+\log\frac{1}{s}\right)^{q\alpha} \frac{ds}{s}, \tag{6.8}$$

with equivalence constants depending on q and α. The upper bound for the left-hand side in terms of the right-hand side in (6.8) follows at once from the weighted Hardy inequality (Proposition 4.6).

As for the lower bound, we will again use a discretization argument and (4.36). Let s_k be as in (6.6) and define $\omega : (0,1) \to [0,\infty)$ as

$$\omega(s) = \frac{1}{s}\left(1 + \log\frac{1}{s}\right)^{q(\alpha-\frac{1}{2})} \quad \text{for } s \in (0,1).$$

Then, by (4.36),

$$\begin{aligned}
\int_0^1 \left(\int_s^1 \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr\right)^q \omega(s)\,ds &\geqslant \sum_{k=1}^{\infty} \int_{s_{k+1}}^{s_k} \left(\int_s^1 \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr\right)^q \omega(s)\,ds \\
&\geqslant \sum_{k=1}^{\infty} \int_{s_{k+1}}^{s_k} \left(\int_{s_k}^{s_{k-1}} \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr\right)^q \omega(s)\,ds \\
&\geqslant \sum_{k=1}^{\infty} f^*(s_{k-1})^q \left(\int_{s_k}^{s_{k-1}} \frac{dr}{r\sqrt{1+\log\frac{1}{r}}}\right)^q \int_{s_{k+1}}^{s_k} \omega(s)\,ds \\
&\geqslant C \sum_{k=2}^{\infty} f^*(s_{k-1})^q \left(1+\log\frac{1}{s_{k-1}}\right)^{\frac{q}{2}} \int_{s_{k-1}}^{s_{k-2}} \omega(s)\,ds \\
&\geqslant C' \sum_{k=2}^{\infty} \int_{s_{k-1}}^{s_{k-2}} f^*(s)^q \left(1+\log\frac{1}{s}\right)^{\frac{q}{2}} \omega(s)\,ds \\
&= C' \int_0^{\frac{1}{2}} f^*(s)^q \left(1+\log\frac{1}{s}\right)^{\frac{q}{2}} \omega(s)\,ds \\
&\geqslant C'' \int_0^1 f^*(s)^q \left(1+\log\frac{1}{s}\right)^{\frac{q}{2}} \omega(s)\,ds,
\end{aligned}$$

for some constants C, C', C'' depending on q and α. This chain yields the lower bound in (6.8). The proof in the case where $1 < q < \infty$ is complete.

Finally, assume that $q = 1$ (and hence $\alpha + 1 < 0$). We first prove the optimality of the range in (6.2). Let $X(\mathbb{R}^n, \gamma_n) = L^{\infty,1;\alpha}(\mathbb{R}^n, \gamma_n)$. Then, by (2.14) and Theorem 4.1, the optimal range $Y(\mathbb{R}^n, \gamma_n)$ for $X(\mathbb{R}^n, \gamma_n)$ fulfils

$$\begin{aligned}
\|f\|_{\overline{Y}'(0,1)} &\approx \left\| \frac{f^{**}(s)}{\sqrt{1+\log\frac{1}{s}}} \right\|_{L^{(1,\infty;-\alpha-1)}(0,1)} \\
&= \sup_{0<s<1} s\left(1+\log\frac{1}{s}\right)^{-\alpha-1} \left(\frac{f^{**}(\cdot)}{\sqrt{1+\log\frac{1}{(\cdot)}}}\right)^{**}(s), \qquad (6.9)
\end{aligned}$$

with equivalence constants depending on α. We claim that

$$\sup_{0<s<1} s\left(1+\log\frac{1}{s}\right)^{-\alpha-1}\left(\frac{f^{**}(\cdot)}{\sqrt{1+\log\frac{1}{(\cdot)}}}\right)^{**}(s) \approx \sup_{0<s<1} f^{**}(s)s\left(1+\log\frac{1}{s}\right)^{-\alpha-\frac{1}{2}}, \tag{6.10}$$

with equivalence constants depending on α. We have that

$$\begin{aligned}
&\sup_{0<s<1} s\left(1+\log\frac{1}{s}\right)^{-\alpha-1}\left(\frac{f^{**}(\cdot)}{\sqrt{1+\log\frac{1}{(\cdot)}}}\right)^{**}(s) \\
&\quad\leqslant \sup_{0<s<1} f^{**}(s)s\left(1+\log\frac{1}{s}\right)^{-\alpha-\frac{1}{2}} \sup_{0<s<1} s\left(1+\log\frac{1}{s}\right)^{-\alpha-1}\frac{1}{s}\left(1+\log\frac{1}{s}\right)^{\alpha} \\
&\quad= \sup_{0<s<1} f^{**}(s)s\left(1+\log\frac{1}{s}\right)^{-\alpha-\frac{1}{2}}.
\end{aligned}$$

The reverse estimate will follow from (6.3) once we show that

$$\begin{aligned}
&\sup_{0<s<1}\left(1+\log\frac{1}{s}\right)^{-\alpha-1}\int_0^s f^*(r)\int_r^s \frac{d\rho}{\rho\sqrt{1+\log\frac{1}{\rho}}}\,dr \\
&\quad\geqslant C\sup_{0<s<1}\left(1+\log\frac{1}{s}\right)^{-\alpha-\frac{1}{2}}\int_0^s f^*(r)\,dr,
\end{aligned} \tag{6.11}$$

for some constant $C=C(\alpha)$. On making use of the same discretization sequence $\{s_k\}$ as in (6.6), we have that

$$\begin{aligned}
&\sup_{0<s<1}\left(1+\log\frac{1}{s}\right)^{-\alpha-1}\int_0^s f^*(r)\int_r^s \frac{d\rho}{\rho\sqrt{1+\log\frac{1}{\rho}}}\,dr \\
&\quad\geqslant \sup_{k\in\mathbb{N}}\sup_{s_{k+1}\leqslant s\leqslant s_k}\left(1+\log\frac{1}{s}\right)^{-\alpha-1}\int_0^s f^*(r)\int_r^s \frac{d\rho}{\rho\sqrt{1+\log\frac{1}{\rho}}}\,dr \\
&\quad\geqslant \sup_{k\in\mathbb{N}}\sup_{s_{k+1}\leqslant s\leqslant s_k}\left(1+\log\frac{1}{s}\right)^{-\alpha-1}\int_0^s f^*(r)\int_r^{s_{k+1}} \frac{d\rho}{\rho\sqrt{1+\log\frac{1}{\rho}}}\,dr \\
&\quad\geqslant \sup_{k\in\mathbb{N}}\sup_{s_{k+1}\leqslant s\leqslant s_k}\left(1+\log\frac{1}{s}\right)^{-\alpha-1}\int_0^{s_{k+2}} f^*(r)\,dr\int_{s_{k+2}}^{s_{k+1}} \frac{d\rho}{\rho\sqrt{1+\log\frac{1}{\rho}}} \\
&\quad\geqslant C\sup_{k\in\mathbb{N}}\sup_{s_{k+1}\leqslant s\leqslant s_k}\int_0^{s_{k+2}} f^*(r)\,dr\left(1+\log\frac{1}{s_{k+1}}\right)^{\frac{1}{2}}\left(1+\log\frac{1}{s_k}\right)^{-\alpha-1}
\end{aligned}$$

$$\geqslant C' \sup_{k\in\mathbb{N}} \sup_{s_{k+3}\leqslant s\leqslant s_{k+2}} \int_0^{s_{k+2}} f^*(r)\,dr\left(1+\log\frac{1}{s_{k+3}}\right)^{-\alpha-\frac{1}{2}}$$

$$\geqslant C' \sup_{k\in\mathbb{N}} \sup_{s_{k+3}\leqslant s\leqslant s_{k+2}} \int_0^{s} f^*(r)\,dr\left(1+\log\frac{1}{s}\right)^{-\alpha-\frac{1}{2}}$$

$$\geqslant C'' \sup_{0<s<1} \left(1+\log\frac{1}{s}\right)^{-\alpha-\frac{1}{2}} \int_0^{s} f^*(r)\,dr,$$

for some constants C, C' and C'' depending on α. This chain implies (6.11). Eq. (6.10) is thus established. Coupling this equation with (6.9) tells us that $\overline{Y}'(0,1) = L^{(1,\infty;-\alpha-\frac{1}{2})}(0,1)$, whence, by (2.14), $Y(\mathbb{R}^n,\gamma_n) = L^{\infty,1;\alpha-\frac{1}{2}}(\mathbb{R}^n,\gamma_n)$.

To prove the optimality of the domain in (6.2), let $Y(\mathbb{R}^n,\gamma_n) = L^{\infty,1;\alpha-\frac{1}{2}}(\mathbb{R}^n,\gamma_n)$. Assumptions (4.5) and (4.6) of Theorem 4.3 are fulfilled by the same reason as in the proof of the case when $1<q<\infty$. Thus, the optimal domain $X(\mathbb{R}^n,\gamma_n)$ for $Y(\mathbb{R}^n,\gamma_n)$ satisfies

$$\|f\|_{\overline{X}(0,1)} \approx \left\| \int_s^1 \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{L^{\infty,1;\alpha-\frac{1}{2}}(0,1)} = \int_0^1 \int_s^1 \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \left(1+\log\frac{1}{s}\right)^{\alpha-\frac{1}{2}} \frac{ds}{s}$$

$$= \int_0^1 \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}} \int_0^r \left(1+\log\frac{1}{s}\right)^{\alpha-\frac{1}{2}} \frac{ds}{s}\,dr \approx \int_0^1 \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}} \left(1+\log\frac{1}{r}\right)^{\alpha+\frac{1}{2}} dr$$

$$= \|f\|_{L^{\infty,1;\alpha}(0,1)},$$

with equivalence constants depending on α. The proof is complete. □

7. Self-optimal spaces

We conclude by exhibiting special r.i. spaces which are self-optimal in the Gaussian Sobolev inequality (1.3). They are the Lorentz endpoint space and the Marcinkiewicz space whose fundamental function is equivalent to $\varphi_G : (0,1) \to [0,\infty)$, given by

$$\varphi_G(s) = e^{-2\sqrt{1+\log\frac{1}{s}}} \quad \text{for } s \in (0,1). \tag{7.1}$$

Theorem 7.1. *Let φ_G be defined by* (7.1).

(i) *There exists an absolute constant C such that*

$$\|u - u_{\gamma_n}\|_{\Lambda^1(\varphi_G')(\mathbb{R}^n,\gamma_n)} \leqslant C\|\nabla u\|_{\Lambda^1(\varphi_G')(\mathbb{R}^n,\gamma_n)} \tag{7.2}$$

for every $u \in V^1\Lambda^1(\varphi_G')(\mathbb{R}^n,\gamma_n)$. Moreover, $(\Lambda^1(\varphi_G')(\mathbb{R}^n,\gamma_n), \Lambda^1(\varphi_G')(\mathbb{R}^n,\gamma_n))$ is an optimal pair in (7.2).

(ii) *There exists an absolute constant C such that*

$$\|u-u_{\gamma_n}\|_{\Gamma^\infty(\varphi_G)(\mathbb{R}^n,\gamma_n)} \leqslant C\|\nabla u\|_{\Gamma^\infty(\varphi_G)(\mathbb{R}^n,\gamma_n)} \tag{7.3}$$

for every $u \in V^1\Gamma^\infty(\varphi_G)(\mathbb{R}^n,\gamma_n)$. Moreover, $(\Gamma^\infty(\varphi_G)(\mathbb{R}^n,\gamma_n), \Gamma^\infty(\varphi_G)(\mathbb{R}^n,\gamma_n))$ is an optimal pair in (7.3).

Note that, in fact,

$$\Gamma^\infty(\varphi_G)\big(\mathbb{R}^n,\gamma_n\big) = \Lambda^\infty(\varphi_G)\big(\mathbb{R}^n,\gamma_n\big) \tag{7.4}$$

(up to equivalent norms), as it is easily seen via the weighted Hardy inequality (Proposition 4.6).

As a consequence of Theorem 7.1(ii), one has the following corollary.

Corollary 7.2. *Let A be a Young function fulfilling* (5.1) *and such that $A(t) = e^{\frac{1}{4}\log^2 t}$ for large t. Then, there exists a constant $C = C(A)$ such that*

$$\|u-u_{\gamma_n}\|_{L^A(\mathbb{R}^n,\gamma_n)} \leqslant C\|\nabla u\|_{L^A(\mathbb{R}^n,\gamma_n)} \tag{7.5}$$

for every $u \in V^1L^A(\mathbb{R}^n,\gamma_n)$. Moreover, $(L^A(\mathbb{R}^n,\gamma_n), L^A(\mathbb{R}^n,\gamma_n))$ is an optimal pair in (7.5).

Corollary 7.2 follows from Theorem 7.1(ii), via the next proposition.

Proposition 7.3. *Let A be a Young function and let $\omega_A : (0,1) \to [0,\infty)$ be the function given by*

$$\omega_A(s) = \frac{1}{A^{-1}(\frac{1}{s})} \quad \text{for } s \in (0,1).$$

If there exists $\delta \in (0,1)$ such that

$$\int_0 A\left(\delta A^{-1}\left(\frac{1}{s}\right)\right) ds < \infty, \tag{7.6}$$

then $L^A(\mathbb{R}^n,\gamma_n) = \Gamma^\infty(\omega_A)(\mathbb{R}^n,\gamma_n)$ (up to equivalent norms).

Proof. Since ω_A is the fundamental function of $L^A(\mathbb{R}^n,\gamma_n)$, the embedding $L^A(\mathbb{R}^n,\gamma_n) \to \Gamma^\infty(\omega_A)(\mathbb{R}^n,\gamma_n)$ is a straightforward consequence of (2.16). On the other hand, since any function u such that $\|u\|_{\Gamma^\infty(\omega_A)(\mathbb{R}^n,\gamma_n)} \leqslant 1$ fulfils the inequality

$$u^*(s) \leqslant A^{-1}\left(\frac{1}{s}\right) \quad \text{for } s \in (0,1), \tag{7.7}$$

the reverse embedding $\Gamma^\infty(\omega_A)(\mathbb{R}^n,\gamma_n) \to L^A(\mathbb{R}^n,\gamma_n)$ follows via (7.6). □

The proof of Theorem 7.1 will make use of the following auxiliary result.

Lemma 7.4. *Let φ and ψ be non-decreasing, weakly differentiable functions on* $[0,1]$ *vanishing at* 0, *and let T be the operator defined by* (4.4).

(i) *If*

$$\sup_{0<s<1} \frac{\int_0^s \sqrt{1+\log\frac{1}{r}}\,\psi'(r)\,dr}{\varphi(s)\sqrt{1+\log\frac{1}{s}}} < \infty, \tag{7.8}$$

then $T : \Lambda^1(\varphi')(0,1) \to \Lambda^1(\psi')(0,1)$. Moreover, the norm of the operator T does not exceed (*an absolute constant times*) *the left-hand side of* (7.8).

(ii) *If*

$$\sup_{0<s\leqslant r<1} \frac{\psi(s)\sqrt{1+\log\frac{1}{s}}}{\varphi(r)\sqrt{1+\log\frac{1}{r}}} < \infty, \tag{7.9}$$

then $T : \Gamma^\infty(\varphi)(0,1) \to \Gamma^\infty(\psi)(0,1)$. Moreover, the norm of the operator T does not exceed (*an absolute constant times*) *the left-hand side of* (7.9).

Proof. (i) One has that $T : \Lambda^1(\varphi')(0,1) \to \Lambda^1(\psi')(0,1)$ if and only if

$$\int_0^1 \left(\sup_{s\leqslant r\leqslant 1} \frac{f^*(r)}{\sqrt{1+\log\frac{1}{r}}} \right) \sqrt{1+\log\frac{1}{s}}\,\psi'(s)\,ds \leqslant C \int_0^1 f^*(s)\,\varphi'(s)\,ds$$

for some constant C and for every $f \in \Lambda^1(\varphi')(0,1)$. Moreover, the optimal constant C in the above inequality equals the norm of T. Thus, the conclusion follows from Proposition 4.7(i).

(ii) Let $f \in M(0,1)$. By (4.32), one has that

$$\begin{aligned}
\sup_{0<s<1} (Tf)^{**}(s)\psi(s) &\leqslant C \sup_{0<s<1} T(f^{**})(s)\psi(s) \\
&\leqslant C \sup_{0<s<1} f^{**}(s)\varphi(s) \sup_{0<s\leqslant r<1} \frac{\psi(s)\sqrt{1+\log\frac{1}{s}}}{\varphi(r)\sqrt{1+\log\frac{1}{r}}},
\end{aligned}$$

for some absolute constant C, and the conclusion follows. $\square$

Now, we are in a position to prove our last main result.

Proof of Theorem 7.1. Throughout the proof, we denote by f an arbitrary function in $\mathcal{M}(0,1)$.

(i) We begin by showing that if $Y(\mathbb{R}^n,\gamma_n) = \Lambda^1(\varphi'_G)(\mathbb{R}^n,\gamma_n)$, then $Y(\mathbb{R}^n,\gamma_n)$ is an optimal domain for itself. The space $Y(\mathbb{R}^n,\gamma_n)$ satisfies assumption (4.5) of Theorem 4.3, since

$$\|f\|_{Y(\mathbb{R}^n,\gamma_n)} = \int_0^1 f^*(s)\frac{e^{-2\sqrt{1+\log\frac{1}{s}}}}{s\sqrt{1+\log\frac{1}{s}}}\,ds \leqslant \sup_{0<s<1}\frac{f^*(s)}{\sqrt{1+\log\frac{1}{s}}}\int_0^1 \frac{e^{-2\sqrt{1+\log\frac{1}{s}}}}{s}\,ds$$
$$\approx \|f\|_{\exp L^2(\mathbb{R}^n,\gamma_n)},$$

with absolute equivalence constants. Assumption (4.6) is also fulfilled, as it is easily seen by an application of Lemma 7.4(ii), since, by (2.17), $(\Lambda^1(\varphi_G'))'(0,1) = \Gamma^\infty(\overline{\varphi}_G)(0,1)$, where

$$\overline{\varphi}_G(s) = \frac{s}{\varphi_G(s)} \quad \text{for } s \in (0,1).$$

Let us now note that

$$\varphi_G'(s) = \frac{\varphi_G(s)}{s\sqrt{1+\log\frac{1}{s}}} \quad \text{for } s \in (0,1), \tag{7.10}$$

and

$$\lim_{s\to 0+}\varphi_G(s) = 0. \tag{7.11}$$

Thus, Theorem 4.3 tells us that the optimal domain $X(\mathbb{R}^n,\gamma_n)$ for $Y(\mathbb{R}^n,\gamma_n)$ obeys

$$\|f\|_{\overline{X}(0,1)} \approx \left\|\int_s^1 \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr\right\|_{\Lambda^1(\varphi_G')(0,1)} = \int_0^1\int_s^1 \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr\,\varphi_G'(s)\,ds$$
$$= \int_0^1 \frac{f(r)}{r\sqrt{1+\log\frac{1}{r}}}\int_0^r \varphi_G'(s)\,ds\,dr = \int_0^1 \frac{\varphi_G(r)}{r\sqrt{1+\log\frac{1}{r}}}f^*(r)\,dr$$
$$= \int_0^1 f^*(r)\varphi_G'(r)\,dr = \|f\|_{\Lambda^1(\varphi_G')(0,1)},$$

with absolute equivalence constants, where the third equality holds by (7.11) and the fourth one by (7.10). This shows that $\Lambda^1(\varphi_G')(\mathbb{R}^n,\gamma_n)$ is the optimal domain for itself.

Now, let $X(\mathbb{R}^n,\gamma_n) = \Lambda^1(\varphi_G')(\mathbb{R}^n,\gamma_n)$. We have to show that $X(\mathbb{R}^n,\gamma_n)$ is the optimal range for itself. By Theorem 4.1 and (2.17), the optimal range $Y(\mathbb{R}^n,\gamma_n)$ for $X(\mathbb{R}^n,\gamma_n)$ satisfies

$$\|f\|_{\overline{Y}'(0,1)} \approx \left\|\frac{f^{**}(s)}{\sqrt{1+\log\frac{1}{s}}}\right\|_{\Gamma^\infty(\overline{\varphi}_G)(0,1)},$$

with absolute equivalence constants. Let $a \in (0,1/2)$. Then

$$\varphi_{Y'}(a) = \|\chi_{(0,a)}\|_{\overline{Y}'(0,1)} \geqslant C \left\| \frac{a\chi_{(a,\frac{1}{2})}(s)}{s\sqrt{1+\log\frac{1}{s}}} \right\|_{\Gamma^\infty(\overline{\varphi}_G)(0,1)}$$
$$= Ca \sup_{0<s<\frac{1}{2}} \overline{\varphi}_G(s) \left(\frac{\chi_{(a,\frac{1}{2})}(\cdot)}{(\cdot)\sqrt{1+\log\frac{1}{(\cdot)}}} \right)^{**}(s),$$

for some absolute constant C. Now,

$$\left(\frac{\chi_{(a,\frac{1}{2})}(\cdot)}{(\cdot)\sqrt{1+\log\frac{1}{(\cdot)}}} \right)^{*}(s) = \chi_{(0,\frac{1}{2}-a)}(s) \frac{1}{(a+s)\sqrt{1+\log\frac{1}{a+s}}} \quad \text{for } s \in (0,1),$$

whence

$$\varphi_{Y'}(a) \geqslant Ca \sup_{0<s<\frac{1}{2}} \frac{\overline{\varphi}_G(s)}{s} \int_0^s \frac{\chi_{(0,\frac{1}{2}-a)}(r)}{(a+r)\sqrt{1+\log\frac{1}{a+r}}}\, dr$$
$$= Ca \sup_{0<s<\frac{1}{2}-a} \frac{\overline{\varphi}_G(s)}{s} \int_0^s \frac{dr}{(a+r)\sqrt{1+\log\frac{1}{a+r}}}\, dr$$
$$= 2Ca \sup_{0<s<\frac{1}{2}-a} e^{2\sqrt{1+\log\frac{1}{s}}} \left(\sqrt{1+\log\frac{1}{a}} - \sqrt{1+\log\frac{1}{a+s}} \right)$$
$$\geqslant 2Ca \sup_{0<s<\frac{1}{2}-a} e^{2\sqrt{1+\log\frac{1}{s}}} \left(\sqrt{1+\log\frac{1}{a}} - \sqrt{1+\log\frac{1}{s}} \right). \tag{7.12}$$

Let s_a be the solution to the equation $\sqrt{1+\log\frac{1}{a}} - \sqrt{1+\log\frac{1}{s_a}} = 1$, namely, $s_a = e^{-(1-2\sqrt{1+\log\frac{1}{a}}+\log\frac{1}{a})}$. Since $\lim_{a\to 0+} s_a = 0$, the number s_a can be used to estimate from below the supremum on the rightmost side of (7.12) when a is sufficiently small. Hence,

$$\varphi_{Y'}(a) \geqslant 2Cae^{2\sqrt{1+\log\frac{1}{a}}-2} = 2Ce^{-2}\overline{\varphi}_G(a) \quad \text{for sufficiently small } a.$$

By (2.16), this entails that $\overline{Y}'(0,1) \to \Gamma^\infty(\overline{\varphi}_G)(0,1)$, whence, by (2.17), $\Lambda^1(\varphi_G')(\mathbb{R}^n,\gamma_n) \to Y(\mathbb{R}^n,\gamma_n)$. Since we already know that inequality (7.2) holds, we conclude that $Y(\mathbb{R}^n,\gamma_n) = \Lambda^1(\varphi_G')(\mathbb{R}^n,\gamma_n)$.

(ii) Let $X(\mathbb{R}^n,\gamma_n) = \Gamma^\infty(\varphi_G)(\mathbb{R}^n,\gamma_n)$. We begin by showing that $X(\mathbb{R}^n,\gamma_n)$ is an optimal range for itself. By Theorem 4.1 and (2.17) such an optimal range $Y(\mathbb{R}^n,\gamma_n)$ fulfils

$$\|f\|_{\overline{Y}'(0,1)} \approx \left\| \frac{f^{**}(s)}{\sqrt{1+\log\frac{1}{s}}} \right\|_{\Lambda^1(\overline{\varphi}_G')(0,1)} \leqslant \int_0^1 \sup_{s\leqslant r\leqslant 1} \frac{f^{**}(r)}{\sqrt{1+\log\frac{1}{r}}} \overline{\varphi}_G'(s)\, ds, \tag{7.13}$$

with absolute equivalence constants. Note that

$$s\int_s^1 \frac{\overline{\varphi}'_G(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \leqslant C\int_0^s \overline{\varphi}'_G(r)\,dr \quad \text{for } s\in(0,1), \tag{7.14}$$

for some absolute constant C, whence by Proposition 4.7(ii),

$$\int_0^1 \sup_{s\leqslant r\leqslant 1} \frac{f^{**}(r)}{\sqrt{1+\log\frac{1}{r}}}\overline{\varphi}'_G(s)\,ds \leqslant C\int_0^1 f^*(s)\overline{\varphi}'_G(s)\,ds, \tag{7.15}$$

for some absolute constant C. From (7.13) and (7.15) we deduce that $\Lambda^1(\overline{\varphi}'_G)(\mathbb{R}^n,\gamma_n)\to Y'(\mathbb{R}^n,\gamma_n)$, and, equivalently, $Y(\mathbb{R}^n,\gamma_n)\to\Gamma^\infty(\varphi_G)(\mathbb{R}^n,\gamma_n)$.

To prove the converse embedding, observe that, since $\overline{\varphi}'_G$ is equivalent to a decreasing function, by the Hardy–Littlewood inequality (2.3)

$$\|f\|_{\overline{Y}'(0,1)} \approx \int_0^1 \left(\frac{f^{**}(\cdot)}{\sqrt{1+\log\frac{1}{(\cdot)}}}\right)^*(s)\overline{\varphi}'_G(s)\,ds \geqslant C\int_0^1 \frac{f^{**}(s)}{\sqrt{1+\log\frac{1}{s}}}\overline{\varphi}'_G(s)\,ds,$$

with absolute equivalence constants and an absolute constant C. Hence, via Fubini's theorem and (7.14),

$$\begin{aligned}\|f\|_{\overline{Y}'(0,1)} &\geqslant C\int_0^1 \frac{\overline{\varphi}'_G(s)}{s\sqrt{1+\log\frac{1}{s}}}\int_0^s f^*(r)\,dr\,ds = C\int_0^1 f^*(r)\int_r^1 \frac{\overline{\varphi}'_G(s)}{s\sqrt{1+\log\frac{1}{s}}}\,ds\,dr\\ &\approx \int_0^1 f^*(r)\overline{\varphi}'_G(r)\,dr = \|f\|_{\Lambda^1(\overline{\varphi}'_G)(0,1)},\end{aligned}$$

with absolute equivalence constants and an absolute constant C. Thus, $Y'(\mathbb{R}^n,\gamma_n)\to\Lambda^1(\overline{\varphi}'_G)(\mathbb{R}^n,\gamma_n)$, and hence $\Gamma^\infty(\varphi_G)(\mathbb{R}^n,\gamma_n)\to Y(\mathbb{R}^n,\gamma_n)$. This shows that the Gaussian Sobolev embedding (7.3) holds, and that the range is optimal.

Now, we shall prove that the domain is optimal as well. To this end, assume that $Y(\mathbb{R}^n,\gamma_n)=\Gamma^\infty(\varphi_G)(\mathbb{R}^n,\gamma_n)$ and note that T is bounded on $\overline{Y}'(0,1)=\Lambda^1(\overline{\varphi}'_G)(0,1)$ by Lemma 7.4(i). Thus, assumption (4.6) is fulfilled. Since an absolute constant C exists such that

$$\varphi_G(s)\leqslant \frac{C}{\sqrt{1+\log\frac{1}{s}}} \quad \text{for } s\in(0,1),$$

one has that $\exp L^2(\mathbb{R}^n,\gamma_n)\to\Lambda^\infty(\varphi_G)(\mathbb{R}^n,\gamma_n)$. Hence, by (7.4), assumption (4.5) holds as well. Therefore, by Theorem 4.3, the optimal domain $X(\mathbb{R}^n,\gamma_n)$ for $Y(\mathbb{R}^n,\gamma_n)$ satisfies

$$\|f\|_{\overline{X}(0,1)} \approx \left\| \int_s^1 \frac{f^*(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{\Gamma^\infty(\varphi_G)(0,1)},$$

with absolute equivalence constants. Since we already know that the Gaussian Sobolev embedding (7.3) holds, and since $X(\mathbb{R}^n, \gamma_n)$ is the largest possible domain when the range is $\Gamma^\infty(\varphi_G)(\mathbb{R}^n, \gamma_n)$, the embedding $\Gamma^\infty(\varphi_G)(\mathbb{R}^n, \gamma_n) \to X(\mathbb{R}^n, \gamma_n)$ certainly holds. What remains to be proved is the reverse embedding, namely that $X(\mathbb{R}^n, \gamma_n) \to \Gamma^\infty(\varphi_G)(\mathbb{R}^n, \gamma_n)$. As a consequence of (2.16), it suffices to show that

$$\varphi_G(a) \leqslant C\varphi_X(a) \quad \text{for } a \in (0,1), \tag{7.16}$$

for some absolute constant C. Given any $a \in (0,1)$, one has that

$$\begin{aligned}
\varphi_X(a) = \|\chi_{(0,a)}\|_{\overline{X}(0,1)} &\approx \left\| \int_s^1 \frac{\chi_{(0,a)}(r)}{r\sqrt{1+\log\frac{1}{r}}}\,dr \right\|_{\Gamma^\infty(\varphi_G)(0,1)} \\
&= \sup_{0<s<1} \frac{\varphi_G(s)}{s} \int_0^s \int_r^1 \frac{\chi_{(0,a)}(\rho)}{\rho\sqrt{1+\log\frac{1}{\rho}}}\,d\rho\,dr \\
&\geqslant \sup_{0<s<a} \varphi_G(s) \int_s^a \frac{d\rho}{\rho\sqrt{1+\log\frac{1}{\rho}}} \\
&= 2 \sup_{0<s<a} \varphi_G(s) \left(\sqrt{1+\log\frac{1}{s}} - \sqrt{1+\log\frac{1}{a}} \right),
\end{aligned} \tag{7.17}$$

with absolute equivalence constants. Let $s_a \in (0,a)$ be the solution to the equation $\sqrt{1+\log\frac{1}{s_a}} - \sqrt{1+\log\frac{1}{a}} = 1$; namely, $s_a = e^{-(1+2\sqrt{1+\log\frac{1}{a}}+\log\frac{1}{a})}$. Then, on making use of s_a to estimate the last supremum in (7.17), we get that

$$\varphi_X(a) \geqslant e^{-2}\varphi_G(a), \tag{7.18}$$

whence (7.16) follows. The proof is complete. □

8. Note added in proof

Recently, the manuscript [28], now published, has been brought to our attention by the authors and independently by the Editors of the JFA. It turns out that the reduction Theorem 3.1 of the present paper coincides with [28, Theorem 3, part (i)]. Our Theorem 3.1 was also announced in [34, Theorem 9.1].

Acknowledgment

We would like to thank the referee for bringing to our attention the paper [13], in which rearrangement inequalities are used to establish optimizing sequences for functional inequalities, and also the paper [26], dealing with Sobolev inequalities in some Orlicz spaces. The results however do not overlap with ours.

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